Galois Cohomology: Some p-adic Hodge Theory

The following are notes from a study group talk I gave in February 2014. They were written primarily for my own benefit before giving the talk, and the results stated fit into a more coherent structure when viewed in step with the rest of the study group!

1. An Introduction to p-adic Hodge Theory

Let $K/\mathbb{Q}_p$ be a finite extension. The study of (finite-dimensional) ℓ-adic Galois representations of the absolute Galois group G_K of K, where $\ell \neq p$, is a well-studied and fruitful area of mathematics. P-adic Hodge theory is the study of ℓ-adic Galois representations for $\ell = p$. On the surface, this is a less accessible topic; there are far more p-adic Galois representations of G_K than ℓ-adic ones, as the topologies of $\mathbb{Q}_p$-vector spaces and on G_K are compatible. The first major conceptual step of p-adic Hodge theory, due to Fontaine, is to isolate nicer subcategories of p-adic representations which are more suitable for study.

As an example, define *weight space* to be the space $\mathcal{W} := \mathrm{Hom}(\mathbb{Z}_p, \mathbb{C}_p^\times)$, which can be viewed as $p-1$ copies of $\mathbb{Z}_p$ living inside $\mathbb{C}_p$. The *cyclotomic character* $\chi : G_K \to \mathbb{Z}_p^\times$ is defined to be the Galois representation that, for a compatible set of p-power roots of unity $(\zeta_{p^n})_n$, takes $g \in G_K$ to $t \in \mathbb{Z}_p^\times$, where $g(\zeta_{p^n}) = \zeta_{p^n}^t$ for all n. Then, for any $s \in \mathcal{W}$, we have a p-adic character χ^s. However, the only 'interesting' characters of this form are those where $s \in \mathbb{Z}$, where here 'interesting' means 'in some sense comes from geometry'. This is made more precise by Fontaine's theory.

Fontaine's main idea was to define *rings of periods*, that is, topological $\mathbb{Q}_p$-algebras $\mathbb{B}$ with a continuous action of G_K. The rings we consider will be *G_K-regular*, and we will assume that $\mathbb{B}^{G_K}$ is a field. The examples we will care about will also carry additional structure (such as a Frobenius operator or a filtration). The mantra here is that for a p-adic representation V and the correct choices of $\mathbb{B}$, the space $\mathbb{D}_{\mathbb{B}}(V) := (V \otimes_{\mathbb{Q}_p} \mathbb{B})^{G_K}$ of G_K-invariants of the tensor product should contain interesting information about V, including inherited structure from $\mathbb{B}$. Indeed, we see that $\mathbb{D}_{\mathbb{B}}(V)$ is a $\mathbb{B}^{G_K}$-vector space, and that

$$\dim_{\mathbb{B}^{G_K}} \mathbb{D}_{\mathbb{B}}(V) \leq \dim_{\mathbb{Q}_p} V.$$

We say that V is *$\mathbb{B}$-admissible* if equality holds here.

We are interested in two particular choices for $\mathbb{B}$. Firstly, the *de Rham* ring of periods, which picks out representations coming from geometrical objects, such as the Tate module of an elliptic curve (see the example below).

Proposition 1.1. *There exists a ring of periods $\mathbb{B}_{\mathrm{dR}}$ and an element $t \in \mathbb{B}_{\mathrm{dR}}$ such that:*

- *(i) For any $g \in G_K$, we have $g(t) = \chi(g)t$, that is, t is a* period *for the cyclotomic character,*
- *(ii) There is a filtration $\mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}}$, indexed by integers, such that $\mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}} = t^i\mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}}$, and*
- *(iii) A p-adic representation V that 'comes from geometry' is $\mathbb{B}_{\mathrm{dR}}$-admissible.*

Remark: The element t is actually a very specific element; however, to define what it is requires a lot more depth than I'm willing to go into here.

Definition 1.2. A $\mathbb{B}_{\mathrm{dR}}$-admissible p-adic representation V is said to be *de Rham.*

Secondly, the *crystalline* ring of periods, which is a subring of $\mathbb{B}_{\mathrm{dR}}$ and picks out de Rham representations coming from objects with good reduction at p.

Proposition 1.3. *There exists a ring of periods $\mathbb{B}_{\mathrm{crys}} \subset \mathbb{B}_{\mathrm{dR}}$ such that:*

(i) There is a natural Frobenius operator φ on $\mathbb{B}_{\mathrm{crys}}$,

(ii) A p-adic representation V that 'comes from geometry with good reduction at p' is $\mathbb{B}_{\mathrm{crys}}$-admissible.

Definition 1.4. A $\mathbb{B}_{\mathrm{crys}}$-admissible p-adic representation V is said to be *crystalline.*

Remark: The rings $\mathbb{B}_{\mathrm{crys}} \subset \mathbb{B}_{\mathrm{dR}}$ are huge. Indeed, they map surjectively onto $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$. They do satisfy the nice property that

$$\mathbb{B}_{\mathrm{dR}}^{G_K} = K \quad \text{and} \quad \mathbb{B}_{\mathrm{crys}}^{G_K} = K_0,$$

where K_0 is the maximal unramified subfield of K. Thus for any p-adic representation V, we know that $\mathbb{D}_{\mathrm{dR}}(V)$ is a K-vector space and $\mathbb{D}_{\mathrm{crys}}(V)$ is a K_0-vector space.

Examples: (i) Let $E/\mathbb{Q}$ be an elliptic curve. Then the *p-adic Tate module* of E is defined to be $T_pE := \lim_{\leftarrow} E[p^n](\overline{\mathbb{Q}})$. This inherits a natural Galois action, and passing to the decomposition group at p, we see that $V_pE := T_pE \otimes_{Zp} \mathbb{Q}_p$ is a 2-dimensional p-adic representation. Indeed, it is a de Rham representation, and if the elliptic curve has good reduction at p, it is crystalline.

(ii) In general, étale cohomology groups are naturally a good source de Rham p-adic representations. To be more precise, let X/K be a proper smooth variety. Then the étale cohomology groups $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ are de Rham. Indeed, this encompasses the first example; the Tate module of an elliptic curve E is the dual of the first étale cohomology group of E. This also gives a more precise notion of 'coming from geometry', that is, a representation that comes from geometry should be isomorphic to an étale cohomology group (or a submodule thereof).

We tend to restrict our study to those p-adic representations that are de Rham, or if we require further structure, those that are crystalline. For further details of the foundations of p-adic Hodge theory, including the construction of the rings $\mathbb{B}_{\mathrm{dR}}$ and $\mathbb{B}_{\mathrm{crys}}$, see [Ber02], which also serves as an excellent guide to further literature on the subject.

2. The Relation to Galois Cohomology

2.1. The Groups $\mathrm{H}^1_*(K, V)$

There is a rather neat connection between these classes of representations and Galois cohomology, which we describe briefly here. Firstly, we give a cute description of the first Galois cohomology group that actually holds in more generality.

Proposition 2.1. *Let $K/\mathbb{Q}_p$ be a finite extension, and let V be a representation of G_K. There is a bijection between isomorphism classes of extensions*

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

of the trivial representation by V and elements of $H^1(K, V)$.

Proof. Suppose we have such an extension, that is, a short exact sequence of Galois modules as given above. By taking Galois cohomology, we obtain a long exact sequence

$$0 \longrightarrow \mathrm{H}^0(K,V) \longrightarrow \mathrm{H}^0(K,E) \longrightarrow \mathbb{Q}_p \xrightarrow{\ \ \delta\ \ } \mathrm{H}^1(K,V).$$

Then define an element $\phi \in \mathrm{H}^1(K,V)$ corresponding to this short exact sequence by $\phi := \delta(1)$. Explicitly, this is obtained as follows; take some element $e \in E$ mapping to 1 in $\mathbb{Q}_p$, and then note that for any $g \in G_K$, the element $g(e) - e$ maps to 0 in $\mathbb{Q}_p$. Thus there is some $v_g \in V$ that maps to $g(e) - e$; we take ϕ to be the class of the cocycle taking $g \mapsto v_g$. Given two isomorphic extensions, passing to the long exact sequences of cohomology of the commutative complex describing the isomorphism shows that the two cocycles we obtain in this manner define the same element of the cohomology.

Conversely, suppose $\phi \in \mathrm{H}^1(K,V)$, and let ψ be a cocyle representing this element. Then define a $\mathbb{Q}_p$-vector space E by $E := V \oplus \mathbb{Q}_p$, and endow this with a G_K action given by

$$\rho_E(g) := \left(\begin{array}{c|c} \rho_V(g) & \psi(g) \\ \hline 0 & 1 \end{array}\right),$$

where $\rho_V : G_K \to \mathrm{GL}(V)$ describes the G_K action on V. It is straightforward, using the cocycle relation on ϕ, to see that this does indeed give a G_K-action as required.

If we choose a different cocycle ψ' representing ϕ, then ψ and ψ' differ by a coboundary, that is, there exists some $a \in V$ such that

$$\psi(g) - \psi'(g) = g(a) - a \quad \forall g \in G_K.$$

If E' is the G_K-representation defined by ψ', we define an isomorphism

$$\begin{aligned} \theta : E &\longrightarrow E', \\ v \oplus \lambda &\longmapsto (v + \lambda a) \oplus \lambda. \end{aligned}$$

It is simple to see that this map is bijective, G_K-equivariant and preserves both the image of V in E and the projection to $\mathbb{Q}_p$. Thus the two extensions E and E' are isomorphic, and we have a well-defined map from $\mathrm{H}^1(K,V)$ to isomorphism classes of extensions.

To see that this association is inverse to the above, we work at the level of cocycles, and we associate to E the cocycle ϕ_E by the method described above. We choose $e = (\ 0 \mid 1\ )^T$, and note that for $g \in G_K$, we have $g(e) - e = (\ \phi(g) \mid 0\ )$. Then $\phi_E(g) := v_g = \phi(g)$, as required. □

In [KB07], this association is used to define certain subgroups of $\mathrm{H}^1(K,V)$ that describe the behaviour of these extensions for suitable V, and give a link between p-adic Hodge theory as described above and the Galois cohomology. To this end, let $K/\mathbb{Q}_p$ be a finite extension, and V a de Rham representation of G_K. Let E be an extension of the trivial representation by V, as above, with associated $\phi \in \mathrm{H}^1(K,V)$, and consider the commutative complex below:

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathbb{Q}_p \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{dR}} \arrow[r] & E \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{dR}} \arrow[r] & \mathbb{B}_{\mathrm{dR}} \arrow[r] & 0. \end{tikzcd}$$

Taking the Galois cohomology of this, we obtain a commutative complex of long exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0(K,V) \arrow[r] \arrow[d] & \mathrm{H}^0(K,E) \arrow[r] \arrow[d] & \mathbb{Q}_p \arrow[r, "\delta"] \arrow[d, "\mathrm{inc}"] & \mathrm{H}^1(K,V) \arrow[r] \arrow[d, "\epsilon"] & \cdots \\ 0 \arrow[r] & \mathbb{D}_{\mathrm{dR}}(V) \arrow[r] & \mathbb{D}_{\mathrm{dR}}(E) \arrow[r, "\beta"] & K \arrow[r, "\gamma"] & \mathrm{H}^1(K, V\otimes \mathbb{B}_{\mathrm{dR}}) \arrow[r] & \cdots . \end{tikzcd}$$

Now, we know that E is de Rham if and only if

$$\dim_K \mathbb{D}_{\mathrm{dR}}(E) = \dim_{\mathbb{Q}_p} E = \dim_{\mathbb{Q}_p} V + 1 = \dim_K \mathbb{D}_{\mathrm{dR}}(V) + 1,$$

that is, if and only if the map β surjects. In this case, the map γ is the zero map, and accordingly, we know that

$$\epsilon(\phi) = \epsilon\circ\delta(1) = \gamma\circ\mathrm{inc}(1) = 0,$$

so that $\phi \in \ker(\epsilon)$. Conversely, if $\phi\in\ker(\epsilon)$, we know that

$$\gamma(1) = \gamma\circ\mathrm{inc}(1) = \epsilon\circ\delta(1) = 0,$$

so that the map γ is the zero map and β surjects. So we've shown that

$$E \text{ is de Rham} \iff \phi\in\ker\Big(\mathrm{H}^1(K,V)\to\mathrm{H}^1(K,V\otimes_{\mathbb{Q}_p}\mathbb{B}_{\mathrm{dR}})\Big).$$

Definition 2.2. Define subspaces $\mathrm{H}^1_*(K,V)$ (for $*=e,f,g$) of $\mathrm{H}^1(K,V)$ as follows:

(i) $\mathrm{H}^1_g(K,V) \coloneqq \ker\Big(\mathrm{H}^1(K,V)\to\mathrm{H}^1(K,V\otimes_{\mathbb{Q}_p}\mathbb{B}_{\mathrm{dR}})\Big)$,

(ii) $\mathrm{H}^1_f(K,V) \coloneqq \ker\Big(\mathrm{H}^1(K,V)\to\mathrm{H}^1(K,V\otimes_{\mathbb{Q}_p}\mathbb{B}_{\mathrm{crys}})\Big)$,

(iii) $\mathrm{H}^1_e(K,V) \coloneqq \ker\Big(\mathrm{H}^1(K,V)\to\mathrm{H}^1(K,V\otimes_{\mathbb{Q}_p}\mathbb{B}_{\mathrm{crys}}^{\varphi=1})\Big)$,

where here $\mathbb{B}_{\mathrm{crys}}^{\varphi=1}$ denotes the subspace of $\mathbb{B}_{\mathrm{crys}}$ fixed by Frobenius.

The following proposition summarises (and extends) the work we did above.

Proposition 2.3. *Let V be a de Rham (resp. crystalline) representation of G_K, and let $0\to V\to E\to\mathbb{Q}_p\to 0$ be an extension of the trivial representation by V corresponding to an element $\phi\in\mathrm{H}^1(K,V)$. Then E is de Rham (resp. crystalline) if and only if $\phi\in\mathrm{H}^1_g(K,V)$ (resp. $\phi\in\mathrm{H}^1_f(K,V)$).*

Remark: In the case $\ell\neq p$, there are similar subgroups; however, we define $\mathrm{H}^1_e(K,V) = 0$ and $\mathrm{H}^1_g(K,V)=\mathrm{H}^1(K,V)$. All of the interesting information is contained therefore in $\mathrm{H}^1_f(K,V)$, which, for V an unramified representation, describes unramified extensions of the trivial representation by V (where here a representation is *unramified* if its kernel contains the inertia subgroup).

2.2. Tate's Duality

Definition 2.4. (New representations from old). For a p-adic representation V, define the *dual representation* V^* by

$$V^* := \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p),$$

with the obvious G_K-action. We also define the *twisted representation* $V(r)$, for $r \in \mathbb{Z}$, by

$$V(r) := V \otimes \chi^r,$$

where χ is the cyclotomic character. Explicitly, this is just endowing V with the G_K-action $\rho_{V(r)}(g) = \rho_V(g) \cdot \chi(g)^r$.

There is a natural pairing $V \times V^*(1) \to \mathbb{Q}_p(1)$ given by evaluation, and Tate showed that cup product for this map gives a perfect pairing

$$\mathrm{H}^1(K, V) \times \mathrm{H}^1(K, V^*(1)) \longrightarrow \mathrm{H}^2(K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p. \tag{1}$$

Proposition 2.5 (Bloch-Kato)**.** *Under the pairing (1), we have:*

(i) $\mathrm{H}^1_e(K, V)$ *and* $\mathrm{H}^1_g(K, V^*(1))$ *are exact annihiliators of each other,*

(ii) $\mathrm{H}^1_f(K, V)$ *and* $\mathrm{H}^1_f(K, V^*(1))$ *are exact annihiliators of each other, and*

(iii) $\mathrm{H}^1_g(K, V)$ *and* $\mathrm{H}^1_e(K, V^*(1))$ *are exact annihiliators of each other.*

Proof. See [KB07], Proposition 3.8. The proof takes up several pages, and contains proofs of some results stated below. ☐

Remark: For the case of the Tate module V_pE of an elliptic curve, we see from this that as $\dim H^1(K, V_pE) = 2$, and $\mathrm{H}^1_e(K, V) \subset \mathrm{H}^1_f(K, V) \subset \mathrm{H}^1_g(K, V)$, we must have $\dim \mathrm{H}^1_*(K, V) = 1$ for $* = e, f, g$, so that in fact the inclusions are equalities. This actually happens fairly often, and for large classes of extensions in which we are interested.

2.3. The Exponential Map

In this section, we describe the definition of the exponential map from [KB07]. We rely on the short exact sequence

$$0 \longrightarrow \mathbb{Q}_p \xrightarrow{\ \alpha\ } \mathbb{B}_{\mathrm{crys}}^{\varphi=1} \oplus \mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}} \xrightarrow{\ \beta\ } \mathbb{B}_{\mathrm{dR}} \longrightarrow 0; \tag{2}$$

unfortunately, we cannot prove the exactness of this sequence without saying a lot more about the structure of the rings involved, but a full description is given in [KB07]. The maps here are $\alpha(x) = (x, x)$ and $\beta(x, y) = x - y$. We also require the following, which is proved in Lemma 3.8.1 of [KB07]:

Lemma 2.6. *Let V be a de Rham representation. The natural map*

$$\mathrm{H}^1(K, \mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}} \otimes V) \longrightarrow \mathrm{H}^1(K, \mathbb{B}_{\mathrm{dR}} \otimes V)$$

is an injection.

After tensoring (2) with V and taking the Galois cohomology, we have a long exact sequence

$$0 \to \mathrm{H}^0(K,V) \to \mathbb{D}_{\mathrm{crys}}^{\varphi=1}(V) \oplus \mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{D}_{\mathrm{dR}}(V) \to \mathrm{H}^1(K,V) \xrightarrow{\ \delta\ }$$
$$\longrightarrow \mathrm{H}^1(K,\mathbb{B}_{\mathrm{crys}}^{\varphi=1}\otimes V) \oplus \mathrm{H}^1(K,\mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}}\otimes V) \to \mathrm{H}^1(K,\mathbb{B}_{\mathrm{dR}}\otimes V) \to \cdots.$$

But by Lemma 2.6, the composition

$$\mathrm{H}^1(K,V) \longrightarrow \mathrm{H}^1(K,\mathbb{B}_{\mathrm{crys}}^{\varphi=1}\otimes V) \oplus \mathrm{H}^1(K,\mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}}\otimes V) \xrightarrow{\mathrm{Proj}} \mathrm{H}^1(K,\mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}}\otimes V)$$

is the zero map, and hence

$$\ker(\delta) = \ker\left(\mathrm{H}^1(K,V) \to \mathrm{H}^1(K,\mathbb{B}_{\mathrm{crys}}^{\varphi=1}\otimes V)\right) = \mathrm{H}^1_e(K,V).$$

By exactness, we see that the map $\mathbb{D}_{\mathrm{dR}}(V) \to \mathrm{H}^1(K,V)$ has image $\mathrm{H}^1_e(K,V)$. We also know exactly what the kernel looks like; indeed, the subset $\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V)$ of $\mathbb{D}_{\mathrm{dR}}(V)$ lies in this kernel, and this allows us to define a surjective map

$$\exp : \mathbb{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V) \longrightarrow \mathrm{H}^1_e(K,V)$$

with kernel $\mathbb{D}_{\mathrm{crys}}^{\varphi=1}/\mathrm{H}^0(K,V)$.

2.4. The Dual Exponential Map

We'd like to define, in some sense, the dual to the exponential map for Tate's pairing. This is done in detail in [Kat93], Section II.1.2; we give the briefest summary.

The cyclotomic character χ gives rise to a continuous function $\log_p(\chi) : G_K \to \mathbb{Z}_p$, and hence an element of $\mathrm{H}^1(K,\mathbb{Z}_p)$. The following proposition is proved in [Kat93]:

Proposition 2.7. *Cup product with* $\log_p(\chi)$ *gives, for each* $i \in \mathbb{Z}$, *an isomorphism*

$$\mathrm{Fil}^i\mathbb{D}_{\mathrm{dR}}(V) = \mathrm{H}^0(K, V\otimes_{\mathbb{Q}_p}\mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}}) \longrightarrow \mathrm{H}^1(K, V\otimes_{\mathbb{Q}_p}\mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}}).$$

The dual exponential map is then the composition

$$\exp^* : \mathrm{H}^1(K,V) \longrightarrow \mathrm{H}^1(K, V\otimes_{\mathbb{Q}_p}\mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}}) \longrightarrow \mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V).$$

It satisfies a suitable compatibility relation under the diagram

$$\begin{tikzcd}
\mathrm{H}^1(K,V) & \times & \mathrm{H}^1(K,V^*(1)) \arrow[r] \arrow[d, "\exp^*"] & \mathbb{Q}_p \\
\mathbb{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V) \arrow[u, "\exp"] & \times & \mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V^*(1)) \arrow[r] & \mathbb{D}_{\mathrm{dR}}(\mathbb{Q}_p(1))/\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(\mathbb{Q}_p(1)),
\end{tikzcd}$$

that is, it is the composition

$$\exp^* : \mathrm{H}^1(K,V) \cong \mathrm{Hom}(\mathrm{H}^1(K,V^*(1)),\mathbb{Q}_p) \longrightarrow \mathrm{Hom}(\mathbb{D}_{\mathrm{dR}}(V^*(1)),\mathbb{Q}_p) \cong \mathbb{D}_{\mathrm{dR}}(V),$$

where the middle map is the dual of the exponential map (considered as a map on $\mathbb{D}_{\mathrm{dR}}(V)$ containing $\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V)$ in its kernel), the first isomorphism comes from Tate's duality, and the second isomorphism comes from the perfect pairing

$$\mathrm{H}^0(K, V\otimes_{\mathbb{Q}_p}\mathbb{D}_{\mathrm{dR}}) \times \mathrm{H}^0(K,(V\otimes_{\mathbb{Q}_p}\mathbb{B}_{\mathrm{dR}})^*(1)) \longrightarrow \mathrm{H}^0(K,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p$$

coming from cup product.

References

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