An Introduction to p-adic L-functions – Exercises III

1. Assessment exercises

Exercise 1. — Show that

$$\overline{\chi}(n)n^{-s} = \frac{(-2\pi i)^s}{\Gamma(s)G(\chi)} \sum_{a\in(\mathbb{Z}/D\mathbb{Z})^\times} \chi(a) \int_{a/D}^{\infty} e^{2\pi i n z}\left(z - \frac{a}{D}\right)^{s-1} dz.$$

Deduce that for $\mathrm{Re}(s) \gg 0$, we have

$$L(f,\overline{\chi},s) = \frac{(-2\pi i)^s}{\Gamma(s)G(\chi)} \sum_{a\in(\mathbb{Z}/D\mathbb{Z})^\times} \chi(a) \int_{a/D}^{\infty} f(z)\left(z - \frac{a}{D}\right)^{s-1} dz.$$

Exercise 2. — Let μ be a locally analytic distribution on $\mathbb{Z}_p$ with growth of order 0. Show that μ is a measure.

Exercise 3. — (1) Show that Γ acts on $\mathbb{A}[r]$ by isometries, that is, that

$$||\gamma \cdot f||_r = ||f||_r.$$

for all $\gamma \in \Gamma$ and $f \in \mathbb{A}[r]$.

(2) Deduce that the function

$$\begin{aligned} ||\cdot||_r : \mathrm{Symb}_\Gamma(\mathcal{D}_k(L)) &\longrightarrow L \\ \Phi &\longmapsto \sup_{D\in\Delta_0} |\Phi(D)|_r \end{aligned}$$

gives a well-defined norm on $\mathrm{Symb}_\Gamma(\mathcal{D}_k(L))$.

(3) Let $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}_k(L))$ satisfy $U_p\Phi = \alpha_p\phi$. Show that for every $D \in \Delta_0$, the distribution $\Phi(D)$ has growth of order $h := v_p(\alpha_p)$.

Exercise 4. — Let $f \in S_{k+2}(\Gamma,\mathbb{C})$ be a cuspidal eigenform with $U_pf = \alpha_pf$. Show that for $0 \leq j \leq k$, we have

$$\begin{aligned} L_p(f,j+1) &:= \int_{\mathbb{Z}_p^\times} z^j \cdot L_p(f) \\ &= \left(1 - \frac{p^j}{\alpha_p}\right) \int_{\mathbb{Z}_p} z^j \cdot L_p(f), \end{aligned}$$

where $(-1)^j = \pm 1$.

Deduce that if f is the modular form attached to an elliptic curve E with split multiplicative reduction at p, then

$$L_p(E,1) := L_p(f,1) = 0,$$

independent of the value of $L(f,1) = L(E,1)$.

Remark: *This vanishing is known as the 'exceptional/trivial zero phenomenon', where the p-adic L-function has a zero arising from its interpolation property. The* Exceptional zero conjecture *of Mazur–Tate–Teitelbaum (Inventiones, 1986), later proved by Greenberg–Stevens (Inventiones, 1993), says that in this case the derivative of $L_p(f)$ at 1 should be equal to*

$$-\mathcal{L}_f \cdot \frac{1}{2\pi i} \cdot \frac{L(f,1)}{\Omega_f^\pm}$$

*where $\mathcal{L}_f$ is an arithmetic $\mathcal{L}$-*invariant *of f. This $\mathcal{L}$-invariant carries deep information about the Galois representation of f, arising in semi-stable p-adic Hodge theory.*

---

2. Additional exercises

Exercise 5. — Define the *Atkin–Lehner* operator W_N on $S_k(\Gamma_0(N))$ by

$$W_N f = i^k N^{1-\frac{k}{2}} f\big|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

(1) Show that W_N is an involution, and thus induces a decomposition $S_k = S_k^+ \oplus S_k^-$.

Suppose now that $f \in S_k(\Gamma_0(N))$ is an eigenform. At level Γ_0, the involution W_N commutes with the Hecke operators, and thus $W_N f = \omega f$ for some $\omega \in \{\pm 1\}$.

(2) Prove that

$$\int_0^1 f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t} = \int_1^\infty (W_N f)\left(\frac{it}{\sqrt{N}}\right) t^{k-s}\frac{dt}{t}.$$

(3) Deduce that:

– the integral defining $L(f,s)$ converges absolutely, and hence that $L(f,s)$ has analytic continuation to the whole complex plane;

– if $\Lambda(s) = \frac{N^{s/2}\Gamma(s)}{(2\pi)^s} \cdot L(f,s)$, then we have the functional equation

$$\Lambda(s) = \omega\Lambda(k-s).$$

Exercise 6. — Let $f \in S_k(\Gamma_0(N),\mathbb{C})$ be a newform, with $p \nmid N$. Consider the forms $f(z), f(pz) \in S_k(\Gamma_0(Np),\mathbb{C})$. Show that the characteristic polynomial of U_p acting on this space is

$$X^2 - a_p(f)X + p^{k-1}.$$

Deduce that the eigenvalues of U_p on $S_k^{p-\mathrm{old}}(\Gamma_0(Np),\mathbb{C})$ all have p-adic valuation $\leq k-1$.

Exercise 7. — Recall the map $\mathrm{Ev}_{\chi,j} : \mathrm{Symb}_\Gamma(V_k(\mathbb{C})) \longrightarrow \mathbb{C}$ from the lecture notes. Prove that

$$\mathrm{Ev}_{\chi,j}(\phi_f) = \binom{k}{j} D^j \cdot \frac{G(\chi)\cdot j!}{(-2\pi i)^{j+1}} \cdot L(f,\overline{\chi},j+1).$$

Exercise 8. — Prove that $\mathbb{Z}_p[[T]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \neq \mathbb{Q}_p[[T]]$.

Exercise 9. — Let $\mathbb{A}(\mathbb{Z}_p, \mathcal{O}_L)$ be the space of rigid analytic functions $\mathbb{Z}_p \to \mathcal{O}_L$. Show that

$$\mathbb{D}(\mathbb{Z}_p,\mathcal{O}_L) \cong \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}(\mathbb{Z}_p,\mathcal{O}_L),\mathcal{O}_L).$$

Show that $\mathscr{M}(\mathbb{Z}_p,\mathcal{O}_L) \subset \mathbb{D}(\mathbb{Z}_p,\mathcal{O}_L)$.

Exercise 10. — Let μ be a rigid analytic distribution on $\mathbb{Z}_p$. Show that μ is locally analytic if and only if its Amice transform $\mathcal{A}_\mu(T)$ converges on the open unit disc in $\mathbb{C}_p$.

Chris Williams, University of Warwick • *E-mail :* christopher.d.williams@warwick.ac.uk