I: THE RATIONAL CASE

I'll start by recalling the rational theory as developed by Pollack and Stevens. The basis for this work is Stevens' control theorem, which is a modular symbol analogue of Coleman's small slope classicality theorem for overconvergent modular forms. First, ~~we need to~~ define modular symbols.

notation: Let p be a prime, 4 $\Gamma = \Gamma_0(N)$ be a congruence subgroup with $p \mid N$. $f \in S_{k+2}(\Gamma)$, $f(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$, $L(f,s) = \sum a_n n^{-s}$

Γ acts on $\mathbb{P}^1(\mathbb{Q})$ by fractional linear transformations, and this action extends naturally to give an action of Γ on $\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$.

For a right Γ-module V, we say a map

$$\phi : \Delta_0 \longrightarrow V$$

is $\underline{\Gamma\text{-invariant}}$ if $\phi(D) = \phi(\gamma D)|\gamma \quad \forall \gamma \in \Gamma$.

Def'n: The space of V-valued modular symbols of level Γ is the space of Γ-inv. functions $\Delta_0 \longrightarrow V$. Denote this space by

$$\mathrm{Symb}_\Gamma(V) := \mathrm{Hom}_\Gamma(\Delta_0, V).$$

This is all very well, but this definition on its own seems a bit unmotivated, and the relation to modular forms isn't immediately clear. To make this connection, we pick some specific spaces V.

Definition: R ring, k integer. Define $V_k(R) :=$ space of homog. polys ~~ref~~ of degree k in two variables. (some right action)

Pollack and Stevens associate to a modular form f a $V_{k-2}(\mathbb{C})$-valued ~~modular~~ harmonic 1-form.

Pollack + Stevens: Let $f \in S_k(\Gamma)$. To f, we associate a $V_{k-2}(\mathbb{C})$-valued harmonic 1-form ω_f, which can then be given $\phi_f \in \mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C}))$ via

$$\phi_f : \{r\} - \{s\} \longmapsto \int_r^s \omega_f .$$

Example: in weight 2, $\omega_f := f(z)\,dz$.

Here there is some right action of Γ on $V_{k-2}(\mathbb{C})$, but I won't define it now; ~~just be~~ instead I'll define it implicitly later on whilst talking about overconvergent symbols.
The action extends to $S_0(p) = \{\left(\begin{smallmatrix} a & \\ c & d \end{smallmatrix}\right) : p | c, \ (a,p)=1\} \circlearrowright V_{k-2}(\mathbb{C}) \leadsto$ Hecke action on $\mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C}))$
This construction, then, associates a modular symbol to a modular form. The following theorem of Eichler and Shimura confirms that these modular symbol spaces really are worth studying:

Theorem: (Eichler-Shimura) There is a Hecke-equivariant isomorphism
$$\mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C})) \cong M_k(\Gamma) \oplus S_k(\Gamma).$$

So this modular symbol space can hopefully tell us quite a lot about the related modular forms. ~~But~~ In passing from a modular form to a modular symbol, we have essentially "thrown away" a lot of the analytic information of the modular form; a modular symbol is an entirely algebraic ~~construct~~ object. We still retain a lot of information, however; in particular, ~~we that~~ the modular symbol "sees" critical values of its L-function.

Example: In weight 2, $L(f,1) = 2\pi i \int_0^\infty f(z)\,dz = \phi_f(0-\infty)$.

In particular, it's reasonable to think that there is some connection to the p-adic L-function. ~~to~~ To find this connection, Stevens introduced overconvergent modular symbols. To define them, I'll have to give a few definitions from p-adic analysis to find spaces to use as coefficient modules.

Def^n: Let $L/\mathbb{Q}_p$ finite. Define $\mathbb{A}_L = \{f(x) = \sum_{n\geq 0} a_n x^n : a_n \in L, \ a_n \to 0\}$.

Let $\Sigma_0(p) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}_p) : ad-bc \neq 0, \ p|c, \ (a,p)=1\}$.
(In particular, note that this group contains the matrices we require for a Hecke action and an action of our congruence subgroup). For a fixed integer k, $\Sigma_0(p)$ acts on $\mathbb{A}$ by

$$\Sigma_0(p) \circlearrowright \mathbb{A} =: \mathbb{A}_k \text{ by } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot f(x) = (a+cx)^k f\left(\tfrac{b+dx}{a+cx}\right).$$

This action makes sense as $p | c$; this is essential! ~~Alone now that~~ We denote this space equipped with this action by $\mathbb{A}_k$.

Note that we can identify V_k with a one variable polynomial space, so

Note: $V_k(L) \subset \mathcal{A}_k(L)$,

$V_k(L)$ preserved by $\Sigma_0(p)$.

In particular we've defined a left-action of $\Sigma_0(p)$ on $V_k(L)$. When we dualise this, we obtain a right action of Γ, as promised.

$\longrightarrow$ right action of $\Sigma_0(p)$ (hence Γ) on $V_k^*(L)$

~~Dua~~ Taking the dual of $\mathcal{A}$ gives us our distribution space:

Def'n: $\mathbb{D}_k(L) = \mathrm{Hom}_{cts}(\mathcal{A}_k(L), L)$;

$\mathbb{D}_k(L) \supseteq \Sigma_0(p)$ by $\mu|\gamma(f) = \mu(\gamma \cdot f)$.

Indeed, dualising the inclusion of V_k into $\mathcal{A}_k$ gives a $\Sigma_0(p)$-equivariant surjection

Fact: $\exists$ $\Sigma_0(p)$-equi. surj.

$$\mathbb{D}_k(L) \twoheadrightarrow V_k^*(L),$$

which then gives a $\Sigma_0(p)$-equivariant surjection

$$\rho: \mathrm{Symb}_\Gamma(\mathbb{D}_k(L)) \longrightarrow \mathrm{Symb}_\Gamma(V_k^*(L)).$$

Def'n: The space of overconvergent mod. symbols is $\mathrm{Symb}_\Gamma(\mathbb{D}_k(L))$.

The map ρ has a huge (infinite-dimensional) kernel.

The main component of Pollack & Stevens' work is the following control theorem, which "controls" the kernel on suitable subspaces

Theorem: (Stevens). Let $\lambda \in L^\times$, $v(\lambda) < k+1$. Then the restriction of ~~ρ to~~ ρ to the λ-spaces of the U_p-operator is an isomorphism:

$$\rho: \mathrm{Symb}_\Gamma(\mathbb{D}_k(L))^{U_p=\lambda} \xrightarrow{\sim} \mathrm{Symb}_\Gamma(V_k^*(L))^{U_p=\lambda}.$$

This should be seen as a modular symbol analogue of Coleman's ~~small~~ classicality theorem, which says that a small slope overconvergent eigenform is in fact classical.

The central theorem ~~leads~~ gives us the p-adic L-function. We start with a cuspidal eigenform f with small slope at p.

$$\begin{tikzcd}
 & & \Psi_f \in \mathrm{Symb}_p(\mathbb{D}_{k-2}) \arrow[d] \\
S_k(\Gamma) \ni f \arrow[rr] & & \phi_f \in \mathrm{Symb}_p(V_{k-2}) \arrow[u, dashed, shift left=6ex]
\end{tikzcd}$$

Theorem: $\mu_f := \Psi_f(0-\infty)\big|_{\mathbb{Z}_p^\times}$ is the p-adic L-function of f.

There is a little extra work to be done here. Firstly, we need to prove that this distribution is admissible, or tempered. This is a fairly formal calculation using the fact that Ψ_f is an eigensymbol. It's straightforward to see that ~~Φ_f~~ μ_f sees the correct L-values. The main part of the work, though, as I said, is in this central theorem.

II: The Bianchi Case

Everything I've said so far is well-known. I'm now going to talk about the case that I've been working on, that is, the case of modular forms over imaginary quadratic fields. These are also known as Bianchi modular forms.

First, some notation:

Let: $K/\mathbb{Q}$ imaginary quadratic, of class number 1, with ring of integers $\mathcal{O}_K$, adele ring $\mathbb{A}_K$. Let p be a rat'l prime.
Let $\mathfrak{n} \subset \mathcal{O}_K$ with $(p) \mid \mathfrak{n}$, $\Gamma := \Gamma_0(\mathfrak{n}) \leq SL_2(\mathcal{O}_K)$ congruence subgroup.

Def'n: A Bianchi modular form of weight $k+2$ and level Γ ~~[illegible]~~ is an adelic automorphic form

$$\Phi : GL_2(\mathbb{A}_K) \longrightarrow V_{2k+2}(\mathbb{C})$$

where here we recall that $V_{k+2}(\mathbb{C})$ is a homogeneous polynomial space.

In the rational case, we can unpick the definition of "adelic automorphic forms" to get the more familiar definition of functions on the upper half plane. In the same way, there is an alternative way of viewing Bianchi modular forms.

Define upper-half-space $\mathcal{H}_3 := GL_2(\mathbb{C})/SU_2(\mathbb{C}) \cong \mathbb{C} \times \mathbb{R}_{>0} = \left\{ \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C},\ t \in \mathbb{R}_{>0} \right\}$.
We can view a Bianchi modular form Φ as a function

$$\mathcal{F} : \mathcal{H}_3 \longrightarrow V_{2k+2}(\mathbb{C})$$

(Note that there is an obvious action of $GL_2(\mathbb{C})$ on $\mathcal{H}_3$).

satisfying ~~suitable transformation properties under~~ Γ.

$$F(\gamma(z,t), \begin{pmatrix} S \\ T \end{pmatrix}) = F((z,t), \begin{pmatrix} \overline{cz+d} & -ct \\ \bar{c}t & cz+d \end{pmatrix}\begin{pmatrix} S \\ T \end{pmatrix}) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

I won't elaborate on the details as to where this comes from; ~~except~~ the details won't be used anywhere in the rest of the talk. Rather than elaborating on the theory, I'll just state some of the relevant properties we'll use about them.

1) There is a good theory of Hecke operators, indexed by the primes of K.

2) A BMF has a Fourier expansion in terms of K-Bessel functions, w/ Fourier coefficients $c(*, \mathcal{F})$ that can be viewed as a function on integral ideals of K.

3) ~~Using~~ we define: $L(\mathcal{F}, s) := \sum_{0 \neq I \subset \mathcal{O}_K} c(I, \mathcal{F}) N(I)^{-s}$.

4) a) (The Clebsch-Gordan formula gives an) injection of $SU_2(\mathbb{C})$-modules
$$V_{2k+2}(\mathbb{C}) \longrightarrow V_k(\mathbb{C}) \otimes V_k(\mathbb{C}) \otimes V_2(\mathbb{C})$$

b) $V_2(\mathbb{C}) \cong \Omega^1(\mathcal{H}_3, \mathbb{C})$ (the space of differential 1-forms on upper half-space is isomorphic to a polynomial space).

And, combining these, we get

c) We can define a $V_k(\mathbb{C}) \otimes V_k(\mathbb{C})$-valued harmonic differential 1-form $\omega_{\mathcal{F}}$ on $\mathcal{H}_3$ associated to $\mathcal{F}$.

This differential allows us to define a modular symbol attached to $\mathcal{F}$.

Def'n: we define $\phi_{\mathcal{F}} : \{r\} - \{s\} \longmapsto \int_r^s \omega_{\mathcal{F}} \in V_k(\mathbb{C}) \otimes V_k(\mathbb{C})$.

It is an elementary check to show that this is a well-defined Γ-invariant function on degree 0 divisors on cusps, and hence we have

Fact: $\phi_{\mathcal{F}} \in \mathrm{Symb}_{\Gamma}(V_k(\mathbb{C}) \otimes V_k(\mathbb{C}))$.

With that in mind, the following definition makes sense.

Def'n: The space of Bianchi Modular Symbols is
$$\mathrm{Symb}_{\Gamma}(V_k^*(\mathbb{C}) \otimes V_k^*(\mathbb{C})).$$

Before moving on to define overconvergent Bianchi modular symbols and look at a control theorem, it's worth mentioning that there is a link between the modular symbol attached to $\mathcal{F}$ and the critical values of twists of its L-function, just as in the rational case. This ~~will give~~ ~~be~~ is necessary if there is to be a connection to p-adic L-functions.

§2.2: Overconvergent Bianchi MS

We've seen that to a BMF $\mathcal{F}$ of weight $k+2$ we can attach a $V_k \otimes V_k$-valued modular symbol. ~~Before~~ The values of this modular symbol can be p-adically interpolated once we've divided by appropriate periods of the modular form, so that we have:

Fact: we can think of $\phi_{\mathcal{F}}$ as living in $\mathrm{Symb}_{\Gamma}(V_k^{(L)} \otimes V_k(L))$, where $L/\mathbb{Q}_p$ is some finite ext.

Now, we can ~~identify~~ embed $V_k \otimes V_k$ ~~with~~ into $\mathbb{A}_k \otimes \mathbb{A}_k$

$$V_k \otimes V_k^{(L)} \subset \mathbb{A}_k^{(L)} \otimes \mathbb{A}_k(L)$$

in much the same way as before. The action of $\Sigma_0(p)$ on $\mathbb{A}_k(L)$ extends in a natural way to give an action of $\Sigma_0(p)^2$ on $\mathbb{A}_k(L) \otimes \mathbb{A}_k(L)$.

Notation: write $V_{k,k}(L) := V_k(L) \otimes V_k(L)$, $\mathbb{A}_{k,k}(L) := \mathbb{A}_k(L) \otimes \mathbb{A}_k(L)$.

Define $\mathbb{D}_{k,k}(L) := \mathrm{Hom}(\mathbb{A}_{k,k}(L), L)$. Then by dualising the inclusion of $V_{k,k}$ into $\mathbb{A}_{k,k}$, we obtain

Fact: $\exists$ a $\Sigma_0(p)^2$-equivariant surjection

$$\pi : \mathbb{D}_{k,k}(L) \longrightarrow V_{k,k}^*(L).$$

It's natural, then, to define

Def'n: The space of overconvergent Bianchi Modular Symbols is

$$\mathrm{Symb}_{\Gamma}(\mathbb{D}_{k,k}(L)).$$

The surjection defined above now gives a surjection from overconvergent symbols to classical symbols, which is the analogue of the specialisation map from the rational case. There is an analogue of the central theorem in this setting, too.

Theorem: (W) Let $\lambda \in L^\times$ satisfy $v(\lambda) < k+1$. Then the restriction of the specialisation map to the λ-eigenspaces of the U_p operator is an isomorphism

$$\mathrm{Symb}_{\Gamma}(\mathbb{D}_{k,k}(L))^{U_p=\lambda} \xrightarrow{\sim} \mathrm{Symb}_{\Gamma}(V_{k,k}^*(L))^{U_p=\lambda}.$$

There is a subtler result in the case that p splits in K.

<u>Theorem</u>: (w) Let $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, Then and let $\lambda_1, \lambda_2 \in L^\times$ with $v(\lambda_1), v(\lambda_2) < k+1$. Then the restriction of the specialisation map

$$\mathrm{Symb}_\Gamma(\mathbb{D}_{k,k}(L))^{U_{\mathfrak{p}}=\lambda_1, U_{\bar{\mathfrak{p}}}=\lambda_2} \xrightarrow{\sim} \mathrm{Symb}_\Gamma(V_{k,k}^*(L))^{U_{\mathfrak{p}}=\lambda_1, U_{\bar{\mathfrak{p}}}=\lambda_2}$$

is an isomorphism.

If there's time, I'll say a quick word about the proof of this theorem, but beforehand, I'll just state some of the consequences. We have similar results about admissibility of the lifted distributions, for which the proof is ~~very~~ just formal manipulation. If we combine this admissibility result with the L-function interpolation property of the classical modular symbol, we have:

<u>Corollary</u>: Let Φ be a small slope Bianchi cuspidal eigenform of weight $k+2$ and level Γ, with associated eigensymbol $\phi_{\mathcal{F}}$. Lift $\phi_{\mathcal{F}}$ to an overconvergent symbol $\Psi_{\mathcal{F}}$. Then

$$\Psi_{\mathcal{F}}(\{0\}-\{\infty\})\big|_{(\mathcal{O}_K\otimes\mathbb{Z}_p)^\times}$$

is the p-adic L-function of $\mathcal{F}$.