AUTOMORPHIC FORMS IN COHOMOLOGY VIA EXAMPLES

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1. Introduction

The aim of today is to give explicit examples of how automorphic forms contribute to the cohomology of locally symmetric spaces (although we will focus on rather concrete examples, and avoid the adelic formalism).

It is hard to overstate the importance of cohomological methods of studying automorphic forms:

(1) We can "forget" the analytic properties of automorphic forms, just using an algebraic theory.

(2) It's much easier to compute explicit examples in this setting (see LMFDB, for example).

(3) We can find Galois representations in the cohomology of locally symmetric spaces.

(4) Using period maps, we can see L-values, and then algebraic structures on cohomology translate into algebraic (and p-adic) properties of L-values. This is fundamental to Iwasawa theory.

(5) There are two main ways of constructing and studying p-adic families of automorphic forms. The first, and original, method uses algebraic structures on locally symmetric spaces that do not exist in general (for example, the case of GL_2 over a non-totally real field). The second is a cohomological approach, via completed cohomology or overconvergent cohomology, which works for general reductive groups.

2. Classical Modular Forms

Fix a congruence subgroup Γ and let $f \in S_{k+2}(\Gamma)$ be a cusp form. There are many different ways of "seeing" f in cohomology.

Definition 2.1. Let $V_k = \mathrm{Sym}^k\,\mathbf{C}^2$ be the space of homogeneous polynomials in two variables X and Y, of degree k. This carries a "natural"[1] action of $\mathrm{GL}_2(\mathbf{C})$.

(1) **de Rham**: Define the differential $\delta_f := f(z)(X - zY)^k dz$ on $\mathcal{H}$, the complex upper half plane. Note that under the action of Γ, $f(z)$ transforms like $(cz + d)^{k+2}$, dz transforms like $(cz + d)^{-2}$ and $(X - zY)^k$ transforms like $(cz + d)^{-k}$, plus the action on X, Y that implies that δ_f descends to a V_k-valued 1-form on $\Gamma\backslash\mathcal{H}$ (we pick an action on V_k, and the polynomial $(X-zY)^k$, so that this would happen). In particular, we get a well-defined class $[\delta_f] =: \theta_f \in H^1_{\mathrm{dR}}(Y_\Gamma, V_k)$.

(2) **compact support**: Let $\Delta_0 := \mathrm{Div}^0(\mathbf{P}^1_{\mathbf{Q}})$ be the space of degree 0 divisors: this can be thought of as the space of "paths between cusps". Let

$$\phi_f : \Delta_0 \to V_k$$

be the map taking $[r] - [s] \mapsto \int_r^s \delta_f$. This is Γ-invariant under the action

$$\phi|_\gamma(D) = \phi(\gamma D)|\gamma,$$

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[1]Quotation marks as the definition varies wildly from paper to paper; the most natural is $(p|\gamma)(x, y)^T = p(\gamma(x, y)^T)$, letting γ act on the column vector $(x, y)^T$. Almost every possible iteration of this appearing as a left- or right-action, with tranposes, adjoints and/or conjugation by finite-order matrices, appears somewhere in the literature...

and we get that $\phi_f \in \mathrm{Hom}_\Gamma(\Delta_0, V_k) =: \mathrm{Symb}_\Gamma(V_k)$. In fact:

Theorem 2.2 (Ash-Stevens, [1])**.**
$$\mathrm{Symb}_\Gamma(V_k) = H^1_c(Y_\Gamma, V_k).$$

For an excellent and down-to-earth introduction to modular symbols, and their use in the study of L-functions, see [7] (and the accompanying lectures by Pollack and Stevens at the Arizona Winter School 2011).

(3) **Group/Singular** Define
$$\psi_f : \Gamma \to V_k, \quad \gamma \mapsto \int_\infty^{\gamma\infty} \delta_f.$$
This is a 1-cocyle, and thus
$$\psi_f \in H^1(\Gamma, V_k) \cong H^1(Y_\Gamma, V_k).$$

In summary, we get maps

$$\begin{tikzcd} & H^1_{\mathrm{dR}} \\ S_{k+2}(\Gamma) \arrow[ur, "f \mapsto \theta_f"] \arrow[r, "f \mapsto \phi_f"] \arrow[dr, "f \mapsto \psi_f"'] & H^1_c \\ & H^1 \end{tikzcd}$$

A natural question to ask: is this map surjective? The answer is no. There are still contributions from anti-holomorphic cusp forms (which one obtains from holomorphic cusp forms via the involution $f(z) \mapsto \overline{f}(z) := f(\overline{z})$, and there are still Eisenstein series (being careful with convergence issues).

Theorem 2.3 (Eichler-Shimura)**.** *The maps above give an Hecke-equivariant isomorphism*
$$S_{k+2}(\Gamma) \oplus \overline{S_{k+2}}(\Gamma) \oplus \mathrm{Eis}_{k+2}(\Gamma) \xrightarrow{\sim} H^1(Y_\Gamma, V_k),$$
where the Hecke operators act on the left via their action on modular forms, and on the right via correspondences.

Proof Sketch. First, one defines a cup product pairing on cohomology, and relates this explicitly to the Petersson inner product on forms. One uses this relation to show injectivity of the map, and then the proof follows from counting the dimensions. (For a complete proof, see [8, §6]). ☐

Remark 2.4 (Warning!)**.** We also have an isomorphism
$$S_{k+2}(\Gamma) \oplus \overline{S_{k+2}}(\Gamma) \oplus \mathrm{Eis}_{k+2}(\Gamma) \xrightarrow{\sim} H^1_c(Y_\Gamma, V_k)$$
to compactly supported cohomology, and there is a natural map $H^1_c \to H^1$, but this is *not* an isomorphism: it has Eisenstein kernel and cokernel. However, the map
$$S_{k+2}(\Gamma) \oplus \overline{S_{k+2}}(\Gamma) \to H^1_c(Y_\Gamma, V_k) \xrightarrow{\sim} H^1(Y_\Gamma, V_k) \xrightarrow{\sim} S_{k+2}(\Gamma) \oplus \overline{S_{k+2}}(\Gamma) \oplus \mathrm{Eis}_{k+2}(\Gamma)$$
induces the identity map on $S_{k+2}(\Gamma) \oplus \overline{S_{k+2}}(\Gamma)$.

In light of this, we make the following definition.

Definition 2.5. Let
$$H^1_{\mathrm{cusp}}(Y_\Gamma, V_k) := S_{k+2}(\Gamma) \oplus \overline{S_{k+2}}(\Gamma) \subset H^1(Y_\Gamma, V_k)$$
denote the image of the cuspidal space in cohomology.

The images of the cusp forms in de Rham, compactly supported or singular cohomology are all canonically isomorphic via Hecke-equivariant maps (so it is just the Eisenstein parts that behave badly).

Note that the work of Venkatesh is not really best studied on the modular forms case: in this case, modular/automorphic forms contribute to just one degree, while Venkatesh's action concerns the "spreading out" of the cohomology in multiple degrees.

3. Bianchi Modular Forms

Classical modular forms are automorphic forms for $\mathrm{GL}_2/\mathbf{Q}$. Bianchi modular forms, on the other hand, are automorphic forms for GL_2/K, where K is an imaginary quadratic field. For simplicity, assume K has class number 1.

3.1. **Motivation.** First, we give some motivation by explaining the above statement for classical modular forms. Note in this case, we have an isomorphism

$$\mathcal{H} \xrightarrow{\sim} \mathrm{GL}_2^+(\mathbf{R})/\mathbf{R}_{\geq 0} \cdot \mathrm{SO}_2(\mathbf{R}),$$

sending $x+iy$ to $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$. In fact, we may define a new function $F : \mathrm{GL}_2^+(\mathbf{R}) \to \mathbf{C}$ by

$$F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (ci+d)^{-k} f\left(\frac{ai+b}{ci+d}\right)$$

This function is naturally left Γ-invariant, but now there is a transformation property under $\mathbf{R}_{\geq 0} \cdot \mathrm{SO}_2(\mathbf{R})$: namely, $F(g\lambda) = \lambda^{-k}F(g)$ for $\lambda > 0$ and $F(gr(\theta)) = e^{ik\theta}$ for $r(\theta) \in \mathrm{SO}_2(\mathbf{R})$.

The point is that F is really a Γ-invariant map of $\mathbf{R}_{>0}\,\mathrm{SO}_2(\mathbf{R})$-representations, and the target is irreducible. In fact, all irreducible algebraic representations of $\mathbf{R}_{>0}\,\mathrm{SO}_2(\mathbf{R})$ are characters, and "weight k" means we are choosing the character $\chi_k : (x, r(\theta)) \mapsto x^{-k}e^{-k\theta}$.

One can lift this further to an adelic interpretation, and the resulting function then generates an automorphic representation, but we will avoid the adelic theory for this talk.

3.2. **The Bianchi Case.**

Remark 3.1. For concreteness, this is all necessarily terse/vague. For precise and complete definitions, largely in the adelic setting, see [5], where the case of general number fields is treated.

We use Section 3.1 to motivate the definition of a Bianchi modular form.

Consider the quotient $\mathrm{GL}_2(\mathbf{C})/\mathbf{C}^\times\,\mathrm{SO}_2(\mathbf{C})$. This is the same as hyperbolic 3-space $\mathcal{H}_3 = \mathbf{C} \times \mathbf{R}_{>0}$ via

$$(z,t) \mapsto \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}.$$

In particular, this is a 3-dimensional real manifold.

So we want to study the algebraic representations of $\mathbf{C}^\times\,\mathrm{SU}_2(\mathbf{C})$.

- The algebraic irreducible representations of $\mathbf{C}^\times$ look like $z \mapsto z^{-k}z^{-\ell}$.
- The algebraic irreducible representations of $\mathrm{SU}_2(\mathbf{C})$ look like $V_n(\mathbf{C}) = \mathrm{Sym}^n(\mathbf{C}^2)$, where the action is given by $p|_u \begin{pmatrix} x \\ y \end{pmatrix} = p\left(u\begin{pmatrix} x \\ y \end{pmatrix}\right)$.

Thus, we will take a *weight* of a Bianchi modular form to be a pair of integers (k,ℓ), which corresponds to choosing the representation $\epsilon_{k,\ell} : \mathbf{C}^\times \to \mathbf{C}^\times$ taking $z \mapsto z^{-k}z^{-\ell}$ and the representation $V_{k+\ell+2}$ of $\mathrm{SU}_2(\mathbf{C})$.

Definition 3.2 (Imprecise)**.** A Bianchi modular form of weight (k,ℓ) and level $\Gamma \subset \mathrm{SL}_2(\mathscr{O}_K)$ is a map of representations $f : \mathcal{H}_3 \to V_{k+\ell+2} \otimes \epsilon_{k,\ell}$ satisfying

- An automorphy condition for Γ (via the action of $\mathbf{C}^\times \mathrm{SU}_2(\mathbf{C})$ on both sides).
- A harmonicity condition.
- A growth condition at the cusps $\mathbf{P}^1_K$ (the boundary of $\mathcal{H}_3$).

We write $M_{k,\ell}(\Gamma)$ for the space of such forms.

Remark 3.3.

- One can show that $M_{k,\ell}(\Gamma)$ is a finite dimensional complex vector space.
- There exist Hecke actions indexed by ideals in $\mathscr{O}_K$.
- As in the classical case, there exist Fourier-Whittaker expansions, and the coefficients are explicitly related to the Hecke eigenvalues for a Bianchi eigenform.
- We can define cusp forms via vanishing of constant Fourier coefficients. Write $S_{k,\ell}(\Gamma)$ for the space of cusp forms.
- We have $S_{k,\ell}(\Gamma) = 0$ when $k \neq \ell$: i.e. all nonzero cusp forms have parallel weight.
- Again all of this is more cleanly defined using the adelic approach, and this then means we can deal with general class number. In the case of class number h, a Bianchi modular form will explicitly be a collection of h functions $\mathcal{H}_3 \to V_{k+\ell+2}$, and if $\mathfrak{p}$ is a non-principal prime, then the Hecke operator $T_\mathfrak{p}$ interchanges these components. This all corresponds to the decomposition of the locally symmetric space (for appropriate level) into connected components indexed by the class group (using strong approximation for $\mathrm{GL}_2(\mathbb{A}_K)$).

3.3. **Differentials on $\mathcal{H}_3$.** Let $\Omega^1(\mathcal{H}_3, \mathbf{C})$ be the space of differential 1-forms on $\mathcal{H}_3$. Note $\mathcal{H}_3 = \mathbf{C} \times \mathbf{R}_{>0}$. One can use this description to show that $\Omega^1(\mathcal{H}_3, \mathbf{C})$ is generated as a $C^\infty(\mathcal{H}_3)$-module by the three elements $dz, d\overline{z}, dt$.

Let $\Omega^2(\mathcal{H}_3, \mathbf{C})$ be the space of differential 2-forms on $\mathcal{H}_3$. This is generated as a $C^\infty(\mathcal{H}_3)$-module by the three elements

$$\frac{dz \wedge d\overline{z}}{t}, \frac{dt \wedge dz}{t}, \frac{dt \wedge d\overline{z}}{t}.$$

These spaces of differential forms are infinite dimensional over $\mathbf{C}$, so we will just extract the part whose coefficients are constants. To that end, let

$$\Omega^1_0(\mathcal{H}_3, \mathbf{C}) := \mathbf{C}dz \oplus \mathbf{C}d\overline{z} \oplus \mathbf{C}dt$$

and

$$\Omega^2_0(\mathcal{H}_3, \mathbf{C}) := \mathbf{C}\frac{dz \wedge d\overline{z}}{t} \oplus \mathbf{C}\frac{dt \wedge dz}{t} \oplus \mathbf{C}\frac{dt \wedge d\overline{z}}{t}.$$

Note $\mathrm{SL}_2(\mathbf{C})$ acts on $\mathcal{H}_3$ by left-translation. This then induces actions on Ω^i_0 for $i = 1, 2$. The key property now is that we can describe these modules very explicitly via spaces we've already seen.

Fact 3.4. *As $\mathrm{SU}_2(\mathbf{C})$-modules, there exist isomorphisms*

$$\begin{aligned} V_2(\mathbf{C}) &\xrightarrow{\sim} \Omega^1_0(\mathcal{H}_3) \\ A^2 &\mapsto dz \\ AB &\mapsto -dt \\ B^2 &\mapsto -d\overline{z} \end{aligned}$$

and

$$\begin{aligned} V_2(\mathbf{C}) &\xrightarrow{\sim} \Omega_0^2(\mathcal{H}_3) \\ A^2 &\mapsto \frac{dt \wedge dz}{t} \\ AB &\mapsto -2\frac{dz \wedge d\overline{z}}{t} \\ B^2 &\mapsto \frac{dt \wedge d\overline{z}}{t} \end{aligned}$$

3.4. **Cohomology Classes Attached to Bianchi Modular Forms.** Let $f \in S_{k,k}(\Gamma)$. This is a map $f : \mathcal{H}_3 \to V_{2k+2}$, as in the definition, satisfying certain properties.

Proposition 3.5 (Clebsch-Gordon)**.** *Without loss of generality, suppose $k \geq \ell$. As $\mathrm{SU}_2(\mathbf{C})$-modules,*

$$V_k \otimes V_\ell \cong V_{k-\ell} \otimes V_{k-\ell-2} \otimes \cdots \otimes V_{k+\ell}.$$

Repeating this several times, we see that $V_k \otimes V_k \otimes V_2$ admits V_{2k+2} as a summand, that is, we get an $\mathrm{SU}_2(\mathbf{C})$-equivariant map

$$V_{2k+2} \hookrightarrow V_k \otimes V_k \otimes V_2.$$

So we can consider f as a map $f : \mathcal{H}_3 \to V_k \otimes V_k \otimes V_2$. Note that this is still $\mathbf{C}^\times \mathrm{SU}_2(\mathbf{C})$-equivariant.

The main idea now is to replace V_2 with Ω_0^1 and Ω_0^2, to obtain δ_f^1 and δ_f^2, which are respectively a $V_k \otimes V_k$-valued 1-form and 2-form on $\mathcal{H}_3$.

Since f is left-Γ-invariant, as in the classical case, δ_f^1 and δ_f^2 descend to forms on $Y_\Gamma = \Gamma\backslash\mathcal{H}_3$. Thus, we get classes

$$\theta_f^i \in H^i_{\mathrm{dR}}(Y_\Gamma, V_k \otimes V_k), i = 1, 2.$$

Definition 3.6. For $i = 1, 2$, let $H^i_{\mathrm{cusp}}(Y_\Gamma, V_k \otimes V_k)$ be the image of the map $S_{k,k}(\Gamma) \to H^i_{\mathrm{dR}}(Y_\Gamma, V_k \otimes V_k)$ taking $f \mapsto \theta_f^i$.

Note that since K has no real embeddings, there is no concept of 'antiholomorphic form' here, and we capture the whole cuspidal space just with $S_{k,k}$. (The mantra is: complex places lead to spreading out across multiple degrees; real places lead to spreading out to higher multiplicity in each degree).

Theorem 3.7 (Eichler-Shimura-Harder, [4, 5])**.** *This gives Hecke-equivariant isomorphisms*

$$S_{k,k}(\Gamma) \xrightarrow{\sim} H^i_{\mathrm{cusp}}(Y_\Gamma, V_k \otimes V_k)$$

for $i = 1, 2$.

Remark 3.8. This provides the 'simplest' setting where Venkatesh's conjecture applies. There are phenomena in this case that the conjecture should allow us to understand better. For example, to a Bianchi modular form f and a prime $\mathfrak{p}$ of K, we can attach two $\mathfrak{p}$-adic $\mathcal{L}$-invariants $\mathcal{L}_f^i$ using the cohomology in degrees $i = 1, 2$. It is natural to speculate that these are equal:

Theorem 3.9 (Gehrman, [2])**.** *If Venkatesh's conjecture is true, then $\mathcal{L}_f^1 = \mathcal{L}_f^2$.*

This is a case where we can obtain potentially different periods arising from the classes in degrees 1 and 2, and Venkatesh can show that they are the same period. There are, however, certain arithmetic periods that arise from only *one* of H^1 or H^1. For example, we can obtain integral formulae for critical values of L-functions attached to f through H^1 [3, 9], but seemingly not through H^2. On the other hand, there exist integral formulae for *non-critical* L-values using H^2 [6]. Wild speculation: can we exploit the structures involved in Venkatesh's conjecture to link this data together, for example through p-adic L-functions?

4. General Number Fields

Let $F/\mathbf{Q}$ be a number field (of class number 1, again for simplicity) of degree $d = r + 2s$, where r and s are as usual. Let $\Sigma = \Sigma(\mathbf{R}) \sqcup \Sigma(\mathbf{C}) \sqcup \overline{\Sigma(\mathbf{C})}$, where $\Sigma(\mathbf{R})$ is the set of real embeddings, $\Sigma(\mathbf{C})$ is a choice of one complex embedding from each pair, and $\overline{\Sigma(\mathbf{C})}$ is the set of conjugate choices.

Let $\mathcal{H}_F = \mathcal{H}_2^r \times \mathcal{H}_3^s$. Then a weight for an automorphic form for GL_2/F should be a d-tuple $\lambda = (k_v)_{v\in\Sigma}$ (plus parity conditions).

Definition 4.1. Let $J \subset \Sigma\mathbf{R}$. An automorphic form for GL_2/F (with respect to J) is basically a holomorphic modular form of weight k_v for each $v \in J$, an antiholomorphic modular form of weight k_v for each $v \in \Sigma(\mathbf{R})\backslash J$, and a Bianchi modular form of weight $(k_v, k_{\overline{v}})$ for each $v \in \Sigma(\mathbf{C})$, neatly packaged into a morphism of representations

$$\mathcal{H}_F \to \bigotimes_{v\in\Sigma(\mathbf{R})} \chi_{k_v} \oplus \bigotimes_{v\in\Sigma(\mathbf{C})} V_{k_v+k_{\overline{v}}+2}.$$

They also have to satisfy an automorphy condition, a harmonicity condition, and a certain growth condition at cusps.

If we impose a vanishing condition on the Fourier series and require that $k_v = k_{\overline{v}}$, we get the space $S_{\lambda,J}(\Gamma)$ of cusp forms of weight λ, holomorphic at J and anti-holomorphic outside J.

To attach a cohomology class, take a subset $J' \subset \Sigma(\mathbf{C})$. Then:

- For all $v \in J$, we get a contribution of dz_v.
- For all $v \in \Sigma(\mathbf{R})\backslash J$, we get a contribution of $d\overline{z_v}$.
- For all $v \in \Sigma(\mathbf{C})\backslash J'$, we get a contribution of a 1-form.
- For all $v \in J'$, we get a contribution of a 2-form.

So the resulting cohomology class has degree $r + s + \#J'$. As J' varies, we get classes in degrees

$$r + s, r + s + 1, \ldots, r + 2s.$$

This exactly matches what James mentioned in the first lecture.

Theorem 4.2. *[5] As J and J' vary, this association gives rise to Hecke-equivariant isomorphisms*

$$\bigoplus_{J\subset\Sigma(R)} \bigoplus_{J'\subset\Sigma(\mathbf{C}),\#J'=t} S_{\lambda,J}(\Gamma) = H^{r+s+t}_{\mathrm{cusp}}(Y_\Gamma, V_\lambda).$$

Note that this gives an explicit realisation of Matsushima's formula in the setting of GL_2 over number fields. In particular, in this case we have $\ell_0 = s = \#\Sigma(\mathbf{C})$. Let the multiplicity with which a cusp form f appears in the lowest degree $r + s$ be D. The isomorphism says that the multiplicity with which a cusp form f appears in the cohomology of degree $r + s + t$ is D times the number of subsets $J' \subset \Sigma(\mathbf{C})$ of size t, of which there are binom(s, t).

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