Adic Spaces Study Group

Easter 2020

Lecture One: Motivation

Chris Lazda

Goal: understand differential/analytic geometry over non-arch. fields, eg $\mathbb{Q}_p$, $\mathbb{C}_p$, $\mathbb{C}((t))$, etc.

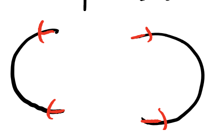
Applications:
- p-adic uniformisation of abelian varieties / curves with "totally degenerate reduction".
- p-adic properties of modular forms (p-towers of modular curves, Rapoport-Zink spaces, eigenvarieties, etc.)
- tropical geometry and mirror symmetry.
- rigid cohomology $\rightsquigarrow$ cohomology groups at "$\ell = p$".

Today: explain the problems that arise in this theory, and why adic spaces are a good fix.

Construction of geometric spaces/manifolds:

local "model" spaces $\rightsquigarrow$ glue together using "allowable" functions $\rightsquigarrow$ global spaces.

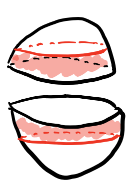
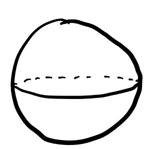
eg. – topological manifolds: local models are open subsets of $\mathbb{R}^n$, gluing: via homeomorphisms between open subsets:

 $\rightsquigarrow S^1 =$ 

– differentiable manifolds: local models are the same; gluing by diffeomorphisms.

– Complex manifolds: local models are open subsets of $\mathbb{C}^n$ (or $B_{\mathbb{C}}^{n,-} = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : |z_i| < 1\}$).

gluing by holomorphic functions on open subsets.

 $\rightsquigarrow$  $\mathbb{P}^1(\mathbb{C})$.

Clear: class of "allowable" functions
$\rightsquigarrow$ gluing operations.

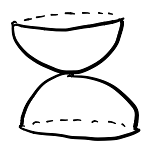
less clear: class of "allowable" functions
$\rightsquigarrow$ structure of local models.

eg. complex analytic spaces; local models: take $f_1, \dots, f_r : B_{\mathbb{C}}^{n,-} \longrightarrow \mathbb{C}$ holomorphic,

$$V = V(f_1, \dots, f_r) = \{ \underline{z} \in B_{\mathbb{C}}^{n,-} : f_i(\underline{z}) = 0 \ \forall i \}.$$

Also get eg.

$$V = \{ (z,w) \in B_{\mathbb{C}}^{2,-} : zw = 0 \};$$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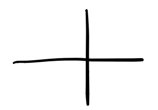


"Allowable" functions: built out of <u>holomorphic</u> function $f : V \to \mathbb{C}$, holomorphic iff $\forall p \in V$, $\exists p \in \mathcal{U} \subseteq B_{\mathbb{C}}^{n,-}$ open s.t.

$$f|_{\mathcal{U} \cap V} = \text{restriction of a holo fn on } \mathcal{U}.$$

Global description: $\mathbb{C}\{z_1, \dots, z_n\}$ = power series convergent on $B_{\mathbb{C}}^{n,-}$,

$$R(V) = \{\text{holo. fns on } V\} \xrightarrow{\simeq} \mathbb{C}\{z_1, \dots, z_n\} \big/ \sqrt{(f_1, \dots, f_r)}. \qquad \text{(radical)}$$

Then $p \in V \rightsquigarrow ev_p : R(V) \longrightarrow \mathbb{C}$,
$m_p := \ker(ev_p)$ f.g. max ideal in $R(V)$.

Then have

$$V \xrightarrow{\simeq} \{\text{f.g. max ideals in } R(V)\},$$
$$p \longmapsto m_p.$$

topology on V: weakest topology s.t. $f : V \longrightarrow \mathbb{C}$ is continuous ($\forall f \in R(V)$), (with the usual topology on $\mathbb{C}$).

eg. 'classical' algebraic geometry $/\mathbb{C}$; local models are affine algebraic sets,

$$f_1, \dots, f_r \in \mathbb{C}[z_1, \dots, z_n],$$
$$V = V(f_1, \dots, f_r) = \{ (z_1, \dots, z_n) \in \mathbb{C}^n : f_i(z) = 0 \ \forall i \}.$$

"Allowable" functions = built out of regular functions $f : V \longrightarrow \mathbb{C}$,

$$R(V) = \{\text{regular functions on } V\} = \mathbb{C}[z_1, \dots, z_n] \big/ \sqrt{(f_1, \dots, f_r)},$$

$$V \xrightarrow{\simeq} \operatorname{maxSpec}(R(V)) = \{\text{max ideals in } R(V)\},$$

Zariski topology = weakest topology s.t. $f: \max\mathrm{Spec}(R(V)) \longrightarrow \mathbb{C}$ is cts $\forall f \in R(V)$, now with the Zariski topology on $\mathbb{C}$;

$\rightsquigarrow$ then glue using regular maps.

Eg. $\mathbb{C} = \max\mathrm{Spec}(\mathbb{C}[z])$ $\qquad$ $\mathbb{C} = \max\mathrm{Spec}(\mathbb{C}[w])$

$$\begin{array}{ccc} \mathbb{C} = \max\mathrm{Spec}(\mathbb{C}[z]) & & \mathbb{C} = \max\mathrm{Spec}(\mathbb{C}[w]) \\ \cup\!\! \mid & & \cup\!\! \mid \\ \mathbb{C}^\times = \max\mathrm{Spec}(\mathbb{C}[z, z^{-1}]) & \xrightarrow[z \mapsto w^{-1}]{\sim} & \mathbb{C}^\times = \max\mathrm{Spec}(\mathbb{C}[w, w^{-1}]), \end{array}$$

and gluing gives the Riemann sphere $\mathbb{P}^1$.

Example: Schemes; A commutative ring,

$$\mathrm{Spec}(A) = \{\text{prime ideals} \subsetneq A\},$$

regular functions on $\mathrm{Spec}(A) = A$. Zariski topology: generated by $D(f) = \{f \notin \mathfrak{p}\}$ $(f \in A)$.

$\Rsh$ + gluing is taken care of by the formalism of locally ringed spaces.

... will see aspects of scheme theory mirrored in adic spaces.

Analytic geometry $\left(/\mathbb{C}_p, \text{ or } \widehat{\mathbb{C}((t))}, \text{ etc} \right)$.

Let $K = \hat{K}$ non-archimedean field; then need:

(i) local model spaces,
(ii) "allowable" functions for gluing. $\qquad$ (+ should see (ii) determines (i)).

1st guess for (i): take

$$B_K^{n,-} = \{(z_1, \dots, z_n) \in K^n, \ |z_i| < 1\},$$

and take

$$f_1, \dots, f_r \in K[\![z_1, \dots, z_r]\!] \quad \text{convergent on } B_K^{n,-},$$
$$V = V(f_1, \dots, f_r) \subset B_K^{n,-}.$$

"exploit clopen"

Simpler technically: K non-archimedean $\Rightarrow$ $B_K^n = \{(z_1, \dots, z_n) \in K^n : |z_i| \le 1\} \subseteq K^n$ open.

Instead, we take

$$f_1, \dots, f_r \in K[\![z_1, \dots, z_r]\!] \quad \text{convergent on } B_K^n,$$
$$V = V(f_1, \dots, f_r) \subset B_K^n.$$

What are the "allowable functions"? Should be analytic in some sense.

... but non-archimedean topologies are totally disconnected...
... so "analyticity" is not a local property.

eg. take $B_K^1 = \mathcal{U} \amalg V$, and $f \equiv 0$ on $\mathcal{U}$, $f \equiv 1$ on V. This cannot globally be defined by a power series.

Now have a choice: use global analyticity, or local analyticity?

If we chose local analyticity: Then can prove $\mathbb{P}^1_K \cong B_K^1$.

... no interesting global geometry...

... and is bad news if we want to study "analytification" of spaces: this process will completely destroy the structure!

Define instead

$$K\langle z_1, \dots, z_n \rangle = \{\text{convergent power series on } B_K^n\}$$
$$= \{\sum a_I z^I \ : \ |a_I| \to 0\}.$$

Tate algebra over K. If $f_1, \dots, f_r \in K\langle z_1, \dots, z_r\rangle$, let

$$A = K\langle z_1, \dots, z_r\rangle / (f_1, \dots, f_r) \qquad \text{affinoid algebra over } K,$$

and

$$\{(z_1, \dots, z_n) \in B_K^n : f_i = 0\} \xrightarrow{\sim} \operatorname{maxSpec}(A)$$
$$P \longmapsto \mathfrak{m}_P := \ker(ev_P).$$

Subspace topology on B_K^n = weakest topology s.t.

$$f : \operatorname{maxSpec}(A) \longrightarrow K \text{ cts}, \qquad \forall f \in A$$

metric topology on K.

$\rightsquigarrow$ mirrors theory of analytic $\mathbb{C}$-spaces nicely.

However: this runs into big problems with gluing. In previous examples, allowable functions are defined using local properties, but our functions are global.

... this is a very serious problem: one reason why it took so long to find a satisfying solution.

2 proposed solutions: * G-topologies: force analyticity to be a local problem *

Tate: (60's). Change the topology on $\operatorname{maxSpec}(A)$; use a coarser topology with fewer open sets and fewer open coverings: G-topology.

... not a topological space: but a categorical gadget like a topology.

... and Tate showed that in the G-topology, analyticity is local.

e.g. in the example above, $U \cup V$ is connected: and we don't get a counterexample.

However: – G-topologies are awkward.
– Stalks no longer detect sheaves (can have all stalks 0, sheaf $\neq 0$).

So this is <u>not</u> the approach we will take.

Huber (90s): We should change our local models.

Replace: $[\mathrm{maxSpec}(A)]$ with $[\mathrm{Spa}(A)]$

valuation spectrum.

Then get:
- honest topological space,
- such that analytic functions are local.

Word of warning: this is a much more <u>radical</u> change than passing from max to prime spectra: but they do provide a very satisfying theory.
more closed pts?

This gives us a <u>huge</u> extra richness in the theory, akin to schemes: we're not restricted to just affinoid algebras any more.

Q: Are the closed pts in the adic spectrum closed? A: <u>No</u>.

Q: Is the adic spectrum somehow a completion? A: Yes, in a sense...

Q: Take care with the structure sheaf is a sheaf! There are non-sheafy cases!

Lecture Two: Rigid Spaces & Formal Schemes

Chris Williams

<u>Recap</u>: K non-archimedean field. We want a "good" theory of analytic geometry over K.

<u>Desirable properties</u>:
1) "GAGA";
2) Integral models ("analytic geometry over $\mathcal{O}_K$").

Successful theories:
1) rigid spaces;
2) formal schemes.

§1: GAGA

<u>Serre</u>: $\exists$ functor

$$\left\{\begin{matrix}\text{schemes loc. of} \\ \text{finite type } /\mathbb{C}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Complex analytic} \\ \text{spaces}\end{matrix}\right\}$$

$$X \longmapsto X^{an},$$

(when X proper)
+ equivalence $\{\text{coherent sheaves on } X\} \simeq \{\text{coherent sheaves on } X^{an}\}$.

(via "allowable functions"): $\rightsquigarrow$ crucially: can recreate X from X^{an}.

(basic construction: closed pts of affine piece are subsets of $\mathbb{C}^n$).

$\rightsquigarrow$ can use techniques from complex analysis / diff geometry in alg. geometry (and vice versa).

Example: $\Gamma \leq SL_2(\mathbb{Z})$, modular curve $Y^{an}_{\Gamma,\mathbb{C}} := \Gamma \backslash \mathcal{H}$ = complex analytic curve

$$\begin{tikzcd}
\text{modular forms} \arrow[dr, squiggly] & & & & \text{p-adic modular forms?} \\
(\mathbb{C}\text{-analytic}) & Y^{an}_{\Gamma,\mathbb{C}} \arrow[d, leftrightarrow, "\text{GAGA}"] & & Y^{an}_{\Gamma,\mathbb{Q}_p} \arrow[ur, squiggly] & \\
(\mathbb{C}\text{-algebraic}) & Y_{\Gamma,\mathbb{C}} \arrow[dr, leftrightarrow] & & Y_{\Gamma,\mathbb{Q}_p} := Y_{\Gamma,\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{Q}_p \arrow[u, dashed] \arrow[dl, leftrightarrow] & \\
& & Y_{\Gamma,\mathbb{Q}} & & \\
& & (\mathbb{Q}\text{-algebraic}) & &
\end{tikzcd}$$

So: want GAGA for analytic spaces over K.

Already seen: "obvious" analogue is very bad.

(non-arch topology $\rightsquigarrow$ totally disconnected $\rightsquigarrow$ too many "locally analytic" functions (allowable)
$\rightsquigarrow$ too many isomorphisms
$\rightsquigarrow$ not enough isomorphism classes!

eg.
$$\begin{tikzcd}
K\text{-schemes} \arrow[r, shift left] & \text{naive } K\text{-an. spaces} \arrow[l, shift left, dashed, "\times"]
\end{tikzcd}$$

§2: Rigid analytic spaces

Chris' talk: introduced rigid analytic spaces.

Recall: Local models: maxSpec(A), $A = K\langle z_1, \dots, z_n \rangle / (f_1, \dots, f_r)$

affinoid algebra.

Chris' talk: big problems with gluing.

$\rightsquigarrow$ must use fiddly notion of G-topology:

"only allow specific kind of open sets/coverings".

Key example: $X = \text{maxSpec}(\mathbb{Q}_p\langle T \rangle)$ = closed rigid disc;

$X(K) = \{ x \in K : |x| \le 1 \}$.

$Y = \text{maxSpec}(\mathbb{Q}_p\langle T, T^{-1} \rangle)$ = unit circle;

$X(K) = \{ x \in K : |x| = 1 \}$.

Z = open rigid disc;

$Z(K) = \{ x \in K : |x| < 1 \}$.

NOT a local model, but infinite union of local models.

In p-adic topology: $X = Y \cup Z$ disconnected.

In rigid topology: Y, Z admissible open sets in X, but $Y \cup Z$ not admissible covering.

$\hookrightarrow$ X is connected!

However...

Theorem: $\exists$ functor

$$\left\{ \begin{matrix} \text{projective schemes} \\ \text{over } K \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{rigid analytic} \\ \text{spaces} / K \end{matrix} \right\},$$

$$X \longmapsto X^{an},$$

+ equivalence of coherent sheaves.

(Kopf?)

$\hookrightarrow$ can build wonderful theory of p-adic (+ overconvergent) modular forms out of rigid analytic modular curves.

Examples: 1) $E/\mathbb{Q}$ elliptic curve. Attached to $E/\mathbb{Q}_p$ have rigid curve $\mathcal{E}/\mathbb{Q}_p$.

Theorem (Tate). Suppose E has multiplicative reduction at p. Then $\exists q \in \mathbb{Q}_p^\times$ s.t.

$$\mathcal{E} \cong \mathbb{G}_m / q^{\mathbb{Z}}$$

over a quadratic extn of $\mathbb{Q}_p$.

$\rightsquigarrow$ "period" q has deep arithmetic interpretations via $\mathcal{L}$-invariants,

(p-adic Hodge theory, exceptional zeros, Iwasawa theory)

2) Rigid analytic modular curve $\xrightarrow{\text{Coleman}}$ p-adic modular forms
(eigencurve, p-adic families).

Upshot: Whilst fiddly to define, rigid analytic spaces have been very successful.

<u>Remarks</u>: (1) For objects "loc. of finite type $/K$", rigid spaces & adic spaces are essentially the same. (equivalence of categories)

(2) Historically, rigid spaces have been easier to work with (more concrete, closer to classical geometry)

(3) ... However: adic spaces have more intuitive geometric properties; e.g. in example above,

(rigid) $\exists$ closed immersion $\underset{\text{open disc}}{Y} \sqcup \underset{\text{circle}}{Z} \xhookrightarrow{\iota} \underset{\text{closed disc}}{X}$, $(Y \sqcup Z)(K) = X(K)$ on pts,
ι <u>not isomorphism</u>
(G-topology, throw out this cover)

(adic) $\exists$ extra points! $\exists$ pt $g \in X(K)$ "between $|x|=1$ and $|x|>1$"
$\rightsquigarrow$ $g \notin Y(K), Z(K)$, $Y \sqcup Z$ not cover (in <u>regular</u> topology)
(+ sheaf property: Chris' talk)

§2: <u>Integral models: formal schemes</u>

Have $\mathrm{Spec}\, \mathbb{Z}_p = \{(0), (p)\}$
$\mathrm{Spec}(\mathbb{Q}_p) \nearrow (0)$, $(p) \hookleftarrow \mathrm{Spec}(\mathbb{F}_p)$

Let $X_{\mathbb{Z}_p}$ scheme over $\mathbb{Z}_p$,
$X_{\mathbb{Q}_p} = X_{\mathbb{Z}_p} \times \mathrm{Spec}\, \mathbb{Q}_p$,
$X_{\mathbb{F}_p} = X_{\mathbb{Z}_p} \times \mathrm{Spec}\, \mathbb{F}_p$.

$$\begin{tikzcd}
X_{\mathbb{Q}_p} \arrow[r, rightsquigarrow] & X^{an}_{\mathbb{Q}_p} & \\
X_{\mathbb{Z}_p} \arrow[u, "\text{gen. fibre}"] \arrow[dr, "\text{spec. fibre}"] & X^{an}_{\mathbb{Z}_p}\,? \arrow[u] \arrow[dr] & \\
& X_{\mathbb{F}_p} & X^{an}_{\mathbb{F}_p}\,?
\end{tikzcd}$$

$\leftsquigarrow$ answer for rigid spaces: <u>formal schemes</u>.

Let A = commutative topological ring, e.g. $\mathbb{C}, \mathbb{Q}_p, \mathbb{Z}_p, \mathbb{Z}_p[[T]]$.

Observation: $[\mathrm{Spec}(A) + \text{Zariski topology}]$ does not "see" the topology on A.

Formal schemes: refinement for special class of topological rings, "adic rings".

("rigid spaces: use topology on K to refine allowable functions. Formal schemes: use topology on rings to refine local models")

Def'n: A commutative ring, $I \subset A$ ideal. The **I-adic topology** is the topology where
$$\{I^n : n \geq 0\} = \text{fundamental basis of nhbds of } 0,$$
ie.
$$\text{Subset } X \subset A \text{ is open} \iff X = \text{union of cosets } a + I^n.$$

A topological ring A is **adic** if $\exists$ ideal $I \subset A$ s.t. topology is the I-adic topology.
↳ say I is an **ideal of definition**.

eg. - $\mathbb{C}$, $\mathbb{Q}_p$ not adic rings with usual topologies.

- $\mathbb{Z}_p$ is an adic ring, $I = (p)$.
- $\mathbb{Z}_p[\![T]\!]$, $I = (p, T)$.
- any A with discrete topology, $I = (0)$.

Def'n: (formal scheme) Let A be an I-adic ring. Define the **formal spectrum** of A to be
$$\mathbf{Spf\ A} := \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{p} \text{ open}\},$$
+ gluing dep. on topology

Basis of open sets
$$D(f) := \{\mathfrak{p} \in \operatorname{Spf} A : f \notin \mathfrak{p}\} \quad \text{for } f \in A,$$

Structure sheaf
$$\mathcal{O}_{\operatorname{Spf} A}(D(f)) = I\text{-adic completion of } A[f^{-1}]$$
$$:= \varprojlim_n A[f^{-1}]/I^n.$$

A formal scheme is a topologically ringed space locally of form $\operatorname{Spf} A$ for an adic ring A.

⟶ we remember the topologies on A!

e.g. - $\operatorname{Spf}(\mathbb{Z}_p) = \{(p)\}$.

- $X = \operatorname{Spf}(\mathbb{Z}_p[\![T]\!])$ = formal open disc over $\mathbb{Z}$; if $K/\mathbb{Q}_p$ non-arch, $\mathcal{O}_K$ = ring of integers.

  Then $X(\mathcal{O}_K) = \mathfrak{m}_K$ max ideal.

- Every scheme is a formal scheme: eg. $\operatorname{Spec} A = \operatorname{Spf}(A, \text{discrete top})$.
  (genuinely enlarged our working space).

<u>Raynauld</u>: Formal schemes over $\mathrm{Spf}\,\mathbb{Z}_p$, p-adic topology, give "good" integral non-arch. geometry.

<u>PROBLEM</u>: $\mathbb{Q}_p$ w/ p-adic topology is <u>not adic</u>! ... $\rightsquigarrow$ $\mathrm{Spf}\,\mathbb{Q}_p$ doesn't make sense;

$\rightsquigarrow$ no obvious "generic fibre"!

<u>Theorem</u>: (Berthelot). $\exists$ a "generic fibre functor"

$$\left\{\begin{matrix}\text{locally finite type}\\ \text{schemes} / \mathrm{Spf}\,\mathbb{Z}_p\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{rigid analytic}\\ \text{spaces} / \mathbb{Q}_p\end{matrix}\right\}$$

$$X \longmapsto X_\eta .$$

<u>Remarks</u>: 1) Further evidence for utility of rigid spaces: formal schemes occur very naturally.

2) Construction is very involved...

3) local models are <u>not</u> sent to local models!!

e.g: formal open unit disc over $\mathbb{Z}_p$ is $X = \mathrm{Spf}\,\mathbb{Z}_p[\![T]\!]$.

Fact: X_η = rigid open unit disc over $\mathbb{Q}_p$.

$$\ldots = \bigcup_{n\geq 1} \mathcal{U}\left(\frac{z^n}{p}\right)$$

$\neq \mathrm{maxSpec}(A)$ for any A !!

<u>§3: Adic reformulation</u>

In world of adic spaces, recover picture "

$$\begin{tikzcd} & \mathrm{Spa}(\mathbb{Z}_p) & \\ \mathrm{Spa}(\mathbb{F}_p) \arrow[ur] & & \mathrm{Spa}(\mathbb{Q}_p) \arrow[ul] \end{tikzcd}$$

"

If X adic $/\mathrm{Spa}(\mathbb{Z}_p)$, can define honest generic fibre $X_\eta := X \times_{\mathrm{Spa}(\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p)$

and have:

$$\begin{tikzcd} \{\text{Formal schemes} / \mathrm{Spf}\,\mathbb{Z}_p\} \arrow[r, "X \mapsto X_\eta"] \arrow[d, "X \mapsto X^{ad}"] & \{\text{rigid spaces} / \mathbb{Q}_p\} \arrow[d, "X \mapsto X^{ad}"] \\ \{\text{Adic spaces} / \mathrm{Spa}(\mathbb{Z}_p)\} \arrow[r, "X^{ad} \mapsto X^{ad}_\eta"] & \{\ \text{ic spaces} / \mathrm{Spa}(\mathbb{Q}_p)\}. \end{tikzcd}$$

LECTURE THREE: HUBER RINGS & ADIC SPECTRA

Rob Rockwood

In this talk:

- We define Huber rings, class of rings containing adic rings, affinoid algebras
- Use to define adic spectrum, topological space underlying local models.

Previous talk: Saw formal schemes and rigid spaces.
(no obvious generic fibres) $\xrightarrow{\text{Berthelot}}$ (no integral models) ← intrinsically

Adic spaces: "nicer" category encompassing both.

Observation: $\mathbb{Q}_p$ w/ p-adic topology is not adic. But: $\mathbb{Q}_p$ contains an open (integral) subring $\mathbb{Z}_p$ which is adic.

Def'n: A topological ring A is Huber if it has an open subring A_0 which is adic with a f.g. ideal of definition. We call such an A_0 a ring of def'n (RoD).

Examples: (1) (Schemes) A discrete ring A is Huber w/ RoD A.
(2) (Formal schemes) A adic w/ f.g. ideal of def'n; then A Huber with RoD A.
(3) (Rigid spaces) A_0 adic, $g \in A_0$ topologically nilpotent ($g^n \to 0$, $n \to \infty$). Then $A = A_0[g^{-1}]$ is Huber.

e.g. $A_0 = \mathcal{O}_K\langle T\rangle$ for K non-arch. field, $A_0[\varpi^{-1}] = K\langle T\rangle$.

Def'n: Let A be a Huber ring. $S \subseteq A$ is bounded if $\forall$ open neighbourhoods $\mathcal{U}$ containing 0, $\exists$ open nbhd $V \ni 0$ s.t. $VS \subseteq \mathcal{U}$.

Lemma: A Huber. $A_0 \subseteq A$ subring is a RoD iff A_0 is open and bounded.

Definition: A Huber ring is Tate if it contains a topologically nilpotent unit $g \in A$. We call such a unit a pseudo-uniformizer.

Proposition: 1) A ring $A = A_0[g^{-1}]$ as above is Tate.
2) Any Tate ring is of this form. (subtleties with g being a zero divisor!)

Example: • K non-arch field, RoI $\mathcal{O}_K$, $g = \varpi \in \mathfrak{m}_K$ any elt of norm <1.
• More generally: A/K algebra, Huber with $\mathfrak{m}$-adic topology; can take the same unit.

Def'n: A Huber ring; $x \in A$ is power bounded if $\{x^n : n \geq 0\}$ is bounded. We write A° for the ring of powerbounded elements.

Examples: – $A = K\langle T\rangle$, then $A^\circ = \mathcal{O}_K\langle T\rangle = A_0$.
– $A = \mathbb{Q}_p[T]/(T^2)$, $A_0 = \mathbb{Z}_p[T]/(T^2)$,
$A^\circ = \mathbb{Z}_p \oplus \mathbb{Q}_p T$.

Proposition: 1) For any ring of def'n A_0, $A_0 \subseteq A^\circ$;
2) A° is the filtered union of the RoD's in A.

(filtered union: $\forall A_0, B_0 \ \exists C_0 \supseteq A_0 \cup B_0$).

Def'n: A Huber. We say A is uniform if A° is bounded; equivalently, A° is a ring of definition.

Def'n: $A^+ \subseteq A$ subring of a Huber ring. A^+ is a ring of integral elements if it is open, integrally closed in A and $A^+ \subseteq A^\circ$. A pair (A, A^+) is called a Huber pair.

Remarks: – We often take $A^+ = A^\circ$.
– topologically nilpotent elements are all in A^+.

§ Continuous Valuations

Similar to Berkovich spaces. Motivation from Gelfand spectrum.

"Γ-norm" ?!

Let Γ be a totally ordered abelian group. A map

$$|\cdot| : A \longrightarrow \Gamma \cup \{0\}$$

totally ordered monoid: $0 < \gamma \ \forall \gamma \in \Gamma$, $0 \cdot \gamma = 0$.

is a continuous valuation if:

(1) $|0| = 0$,
(2) $|1| = 1$,
(3) $|xy| = |x||y| \quad \forall x, y \in A$,
(4) $|x+y| \leq \max\{|x|, |y|\}$,
(5) $\forall \gamma \in$ Image $|A|$, the set $\{a \in A : |a| < \gamma\}$ is open. (continuity)

Say two continuous valuations $|\cdot|, |\cdot|'$ are equivalent if $|a| \geq |b|$ iff $|a|' \geq |b|' \quad \forall a, b$.

Def'n: Given a Huber pair (A, A^+), we define the adic spectrum

$$\mathrm{Spa}(A, A^+) := \{ |\cdot| : A \to \Gamma_{|\cdot|} \cup \{0\} \text{ cts valuations} : |A^+| \leq 1 \} / \sim.$$

For $f, g \in A$, define

$$\mathcal{U}(f/g) := \{ x \in \mathrm{Spa}(A, A^+) : |f(x)| \leq |g(x)| \neq 0 \}.$$

Here $|f(x)|$ means: choose a rep've $|\cdot|_x$ of x; then $|f|_x = |f(x)|$. Order is preserved by definition. The sets $\mathcal{U}(f/g)$ are called rational subsets.

Remark: $\{x : |f(x)| \neq 0\}$, $\{x : |f(x)| \leq 1\}$ are rational subsets.

Def'n: A topological space X is spectral if ∃ a ring R s.t. $X \simeq \mathrm{Spec}(R)$.

Theorem: The adic spectrum is spectral.

(Minhyong: this is a bit of a disappointment; it's not true of Berkovich spaces)

Examples: ① $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$, $\mathbb{Z}$ with discrete topology. We have 3 types of points:

a) $\eta : \mathbb{Z} \longrightarrow \{0, 1\}$, sending non-zero integers to 1.
b) p prime, $\eta_p : \mathbb{Z} \longrightarrow \mathbb{F}_p \longrightarrow \{0, 1\}$, i.e. $\eta_p(a) = 1 \Leftrightarrow p \nmid a$.
c) p prime, $s_p : \mathbb{Z} \longrightarrow \mathbb{Z}_p \xrightarrow{v_p} p^{\mathbb{Z}_{\leq 0}} \cup \{0\}$.

Note that $\overline{\{\eta\}} = \mathrm{Spa}(\mathbb{Z},\mathbb{Z})$; $\overline{\{\eta_p\}} = \{\eta_p, s_p\}$; s_p is a closed point.

② K non-archimedean, algebraically closed, spherically complete (descending intersections of closed balls are non-empty), value group $\mathbb{R}_{>0}$. We get 3 types of point in $\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$:

↖ closed unit disc.

a) $x \in K$, $|x| \le 1$ gives $x : f \longmapsto |f(x)|_K$.

b) $x \in \mathcal{O}_K$, $r \in (0,1]$, gives

$$x_r : f \longmapsto \sup_{y \in B(x,r)} |f(y)|,$$

(Gauss norm: multiplicativity is not obvious)

$B(x,r) = \{ y \in K : |x-y| \le r \}$.

c) (Rank 2 pt). $x \in \mathcal{O}_K$, $r \in (0,1]$. $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, $\gamma > 1$, ordered lexicographically ("infinitesimally between elements of $\mathbb{R}_{>0}$"),

$$x_r^{\pm} : f = \sum a_n (T-x)^n \longrightarrow \max_n (|a_n| r^n, \gamma^{\pm n})$$

Note: $x_{1^+} \notin \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$.

The rank 2 points connect the closed unit disc:

Define $S^1_{ad} = \{ |T| = 1 \}$, $\mathcal{U} = \bigcup_{\varepsilon > 0} \{ |T| \le 1-\varepsilon \}$:

and $\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle) \setminus (S^1_{ad} \cup \mathcal{U}) \ni x_{1^-}$, as $x_1^-(T) = (1, \gamma^{-1}) > 1-\varepsilon$ $\forall \varepsilon$, but < 1!

Remark: (David). Type (b) points correspond to taking suprema of power series coefficients. Type (c) points additionally remember either the first (−) or last (+) time this supremum is obtained.

Remark: (Diane). The Berkovich closed disc is connected, but doesn't have the rank 2 points. Chris L: this is not breaking the example, since S^1_{Berk} is not open in the closed disc.

... so why adic? They give nice topological explanations of rigid behaviour.

LECTURE FOUR: GLOBAL ADIC SPACES

David Loeffler

Last time:
- adic rings = topological rings with I-adic topology, some ideal I.
- Huber ring: top ring w/ open adic subring.
- Huber pair: (A, A^+), A Huber ring, A^+ open subring, int. closed + powerbounded.

Note: Huber pairs are a category:

$$\mathrm{Hom}\big((A,A^+), (B,B^+)\big) = \{ f : A \to B \; : \; f(A^+) \subset B^+ \}.$$

Then

$$\mathrm{Spa}(A, A^+) = \{ \text{equiv. classes of cts valuations on } A, \le 1 \text{ on } A^+ \}.$$

Abuse of notation: $|\cdot(x)|$ for elt of Spa, ie. $f \longmapsto |f(x)|$.

(... but there is no x!)

§1: More on Spa

Proposition: 1) Spa is a contravariant functor
$$(\text{Huber pairs}) \longrightarrow \underline{\text{Top}}$$
(need to check continuity).

2) If $\hat{A}$ = (separated) completion of A, then
$$\mathrm{Spa}(\hat{A}, \hat{A}^+) = \mathrm{Spa}(A, A^+).$$
($\hat{A} = \varprojlim A/I^n$).

↳ So: can always assume A is complete.

Proof: easy.

Proposition: (much deeper). Let (A, A^+) complete. Then:

(here: complete means separated completion).

1) If $A \neq \{0\}$, then $\mathrm{Spa}(A, A^+) \neq \emptyset$.

2) $A^+ = \{ f \in A : |f(x)| \leq 1 \quad \forall x \in \mathrm{Spa} \}$

3) $f \in A^\times \iff |f(x)| \neq 0 \quad \forall x \in \mathrm{Spa}$.

(all quite fiddly: see Huber '93).

§2: Rational subsets

Recall: the topology on $\mathrm{Spa}(A, A^+)$ is the coarsest one such that
$$\mathcal{U}(f/g) = \{ x : |f(x)| \leq |g(x)| \neq 0 \} \qquad \forall f, g \in A.$$
warning: $\mathcal{U}(af/ag) \neq$ in general!

Definition: A rational subset is a set of the form
$$\mathcal{U}(t_1/s) \cup \cdots \cup \mathcal{U}(t_n/s),$$
where $T = \{t_1, \ldots, t_n\}$ generate an open ideal.

E.g. In $\mathrm{Spa}(\mathbb{Q}_p\langle x\rangle, \mathbb{Z}_p\langle x\rangle)$:

- $\mathcal{U}(p/x)$ is a rational subset ($pA = A$);
- $\mathcal{U}(x/p)$ doesn't look rational (xA not open), but
$$\mathcal{U}(x/p) = \mathcal{U}\left(\frac{\{p, x\}}{p}\right) = \mathcal{U}\left(\frac{p}{p}\right) \cup \mathcal{U}\left(\frac{x}{p}\right),$$
so it is rational!
- $\mathcal{U}(0/x)$ is not rational.

Geometrically this is removing $x = 0$ from the closed disc: not quasi-compact ("leaks around the hole at $x=0$").

"$\exists!$ way of making $\mathcal{U}$ into an adic space in its own right"

Theorem: (Huber). Let $\mathcal{U} = \mathcal{U}(\frac{T}{s})$ rational subset. Then $\exists!$ morphism
$$f_\mathcal{U} : (A, A^+) \longrightarrow (A_\mathcal{U}, A_\mathcal{U}^+)$$

of complete Huber pairs such that:

- $f_\mathcal{U}^*(\mathrm{Spa}(A_\mathcal{U}, A_\mathcal{U}^+)) \subseteq \mathcal{U}$,
- $f_\mathcal{U}$ is universal with this property: for any $f : (A, A^+) \longrightarrow (B, B^+)$ s.t. $f^*(\mathrm{Spa}(B,B^+)) \subset \mathcal{U}$, then f factors uniquely through $f_\mathcal{U}$.

Moreover $f_\mathcal{U}^*$ is a homeomorphism $\mathrm{Spa}(A_\mathcal{U}, A_\mathcal{U}^+) \simeq \mathcal{U}$.

Idea of the proof: $A[s^{-1}]$ localisation of A, $A[T/s]$ = subring generated by $\{t/s : t \in T\}$. (Must be careful: s could be a 0-divisor, and $A \not\hookrightarrow A[s^{-1}]$). Then $A_\mathcal{U}$ = completion in I-adic topology, some ideal of def'n I. $\hookrightarrow$ "$A_\mathcal{U} = A\langle \frac{T}{s} \rangle$", in rigid language.

Then $A_\mathcal{U}^+$ = integral closure of $A^+[T/s]$ in $A_\mathcal{U}$.

... but lots more required: proved in detail in Scholze-Weinstein.

Examples: $\mathrm{Spa}(\mathbb{Q}_p\langle x\rangle, \mathbb{Z}_p\langle x \rangle)$.

① $\mathcal{U} = \mathcal{U}\left(\frac{\{p, x\}}{p}\right)$, $A_\mathcal{U} = \mathbb{Q}_p\langle \frac{x}{p} \rangle = \{\sum a_n x^n \in \mathbb{Q}_p[\![x]\!] : a_n p^n \to 0\}$

② $\mathcal{U} = \mathcal{U}\left(\frac{p}{x}\right)$: $A_\mathcal{U} = \{\sum_{n=-\infty}^{\infty} a_n x^n : \text{cvgt for } |p| \le |x| \le 1\}$ (annulus).

Here:

① is "$\{x : |x| \le |p|\}$" inside $\{x : |x| \le 1\}$,

② is "$\{x : |p| \le |x| \le 1\}$".

§3: The structure presheaf

Definition: For $W \subset \mathrm{Spa}(A, A^+)$ open, define

$$\mathcal{O}(W) = \varprojlim_{\mathcal{U} \subset W \text{ rat'l}} A_\mathcal{U}, \qquad \text{(structure presheaf)}$$

$$\mathcal{O}^+(W) = \varprojlim_{\mathcal{U} \subset W \text{ rat'l}} A_\mathcal{U}^+. \qquad \text{(Integral structure presheaf)}$$

Lemma: Every open subset is a union of rational subsets. (c.f. Morel's lecture notes).

$\hookrightarrow$ serious amount of work: "rat'l subsets are a basis for the topology"

Proof crucially uses fact that ideals of def'n are f.g.: first place this comes in.

(without this, the structure presheaf might not be able to tell open sets apart!)

This defines two presheaves on $\mathrm{Spa}(A, A^+)$.

Definition: We say that (A, A^+) is sheafy if $\mathcal{O}(-)$ is a sheaf. ($\Rightarrow \mathcal{O}^+(-)$ is also a sheaf).

Theorem: (A, A^+) is sheafy if:

1) A is discrete (case of schemes). ⇝ recover usual structure sheaf.
2) A is f.g. over a Noetherian ring of definition (formal schemes, rigid geometry / discretely valued fields)
3) A is Tate (see previous lecture) and $A\langle T_1, \ldots, T_n\rangle$ is Noetherian $\forall n$. (rigid geometry / $\mathbb{C}_p$).

(... and perfectoid spaces!)

§4: Adic spaces

Let $\mathcal{V}$ = category of "valued ringed spaces", with

$$\mathrm{Obj}(\mathcal{V}) = \{(X, \mathcal{O}_X, |\cdot(x)|_{x\in X}) : X \text{ topological space}, \mathcal{O}_X \text{ sheaf of rings}, |\cdot(x)| \text{ equiv. class of cts valuations on } \mathcal{O}_{X,x}\},$$

+ obvious notion of morphism.

Clearly Spa is a functor (Sheafy Huber pairs) $\longrightarrow \mathcal{V}$, contravariant.

Definition: An adic space is an object of $\mathcal{V}$ that has a covering by Spa of sheafy Huber pairs.
(no need for this to be finite, or countable, cover!)

+ $\exists$ more general notion of pre-adic space, incorporating non-sheafy Huber pairs.
... but this is a nightmare to work with: gluing doesn't work without sheafiness!

Lecture Five: Topology and Examples

Pak-Hin Lee

Consider $X = \mathrm{Spa}(\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!])$ with (p,T)-adic topology.

- $\mathbb{Z}_p[\![T]\!]$ is complete regular local noetherian of dim 2 (but not Tate).
- hence X is sheafy.

Points of X:

- unique pt with open kernel (or support)
$$x_{\mathbb{F}_p} : \mathbb{Z}_p[\![T]\!] \longrightarrow \mathbb{F}_p \longrightarrow \{0,1\},$$
second map sending $\mathbb{F}_p^\times \to 1$.

- remove the closed point $x_{\mathbb{F}_p}$, and define open adic subspace
$$\mathcal{Y} = X \setminus \{x_{\mathbb{F}_p}\}.$$
Then all points of $\mathcal{Y}$ have non-open kernel. Such points are called analytic.

We can think of p and T as co-ordinate functions on X. In this perspective, "$p=0$" is the horizontal axis and "$T=0$" the vertical one. On $\mathcal{Y}$, they cannot both be zero.

- The locus "$T=0$" consists of valuations factoring through (in the rank 1 case)

$$\mathbb{Z}_p[[T]] \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{R}_{\geq 0},$$

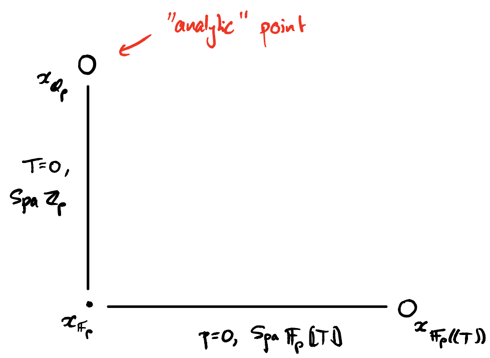
ie. two points:
- closed point $x_{\mathbb{F}_p}$,
- and a generic point $x_{\mathbb{Q}_p}$ $(\sim \operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p))$.

- Locus "$p=0$" consists of valuations factoring through

$$\mathbb{Z}_p[[T]] \longrightarrow \mathbb{F}_p[[T]] \longrightarrow \mathbb{R}_{\geq 0},$$

ie. two points:
- closed point $x_{\mathbb{F}_p}$,
- generic pt $x_{\mathbb{F}_p((T))}$ $(\sim \operatorname{Spa}(\mathbb{F}_p[[T]], \mathbb{F}_p[[T]]))$.

The picture is:



All other points of $\mathcal{Y}$ lie somewhere in this first quadrant. This can be measured by:

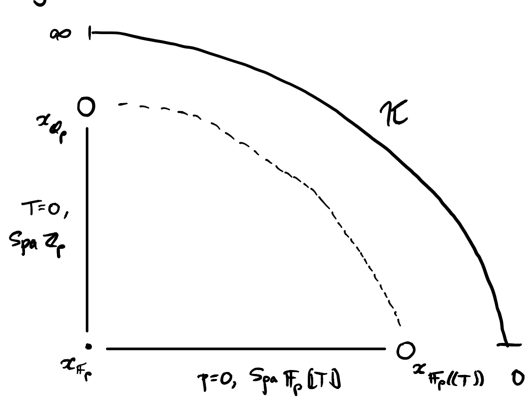
Proposition: There is a unique continuous map

$$\kappa : |\mathcal{Y}| \longrightarrow [0, \infty]$$

such that for $y \in \mathcal{Y}$:

- $|T(y)|^n \geq |p(y)|^m$ for all $\frac{m}{n} > \kappa(y)$,
- $|T(y)|^n \leq |p(y)|^m$ for all $m/n < \kappa(y)$.

Extreme cases:
- $\kappa = 0$ along "$p=0$", ie. the horizontal axis.
- $\kappa = \infty$ along "$T=0$", the vertical axis.



Example: every $z \in \mathbb{C}_p$ with $|z|_p < 1$ (in the p-adic abs. value) defines a valuation on $\mathbb{Z}_p[\![T]\!]$:

$$|\cdot|_z \; : \; f \longmapsto |f(z)|_p.$$

Then

$$\kappa(|\cdot|_z) = v_p(z).$$

For an interval $I \subset [0,\infty]$, define

$$\mathcal{Y}_I := \kappa^{-1}(I).$$

A non-affinoid "generic fibre": Have $\mathcal{Y}_{(0,\infty]}$ is the complement of "$p=0$", i.e. the generic fibre of $X = \operatorname{Spa} \mathbb{Z}_p[\![T]\!]$ over $\operatorname{Spa} \mathbb{Z}_p$.

... but $\kappa(\mathcal{Y}_{(0,\infty]}) = (0,\infty]$ is not compact. Thus $\mathcal{Y}_{(0,\infty]}$ is not quasicompact, and cannot be affinoid!

This is happening because there is no sensible Huber ring structure on $\mathbb{Z}_p[\![T]\!][\frac{1}{p}]$, which would be the natural "generic fibre" to consider. There is an additional subtlety: the construction of fibre products of adic spaces is subtle; the map

$$\mathbb{Z}_p \longrightarrow \mathbb{Z}_p[\![T]\!]$$

is not adic!

As in the rigid world, we can cover this with an infinite union of rational sets. In particular,

$$\mathcal{Y}_{(0,\infty]} = \bigcup_{n \geq 1} \mathcal{Y}_{[\frac{1}{n},\infty]},$$

and each $\mathcal{Y}_{[\frac{1}{n},\infty]} = \{|T| \leq |p|^{1/n}\}$ rational. More precisely, we are considering $\mathbb{Z}_p[\![T]\!][\frac{T^n}{p}]$; we must complete, and then find

$$\mathcal{Y}_{[\frac{1}{n},\infty]} = \operatorname{Spa}\left(\mathbb{Q}_p\langle T, \tfrac{T^n}{p}\rangle, \; \mathbb{Z}_p\langle T, \tfrac{T^n}{p}\rangle\right).$$

A strange neighbourhood of $x_{\mathbb{F}_p((T))}$:

Consider $\mathcal{Y}_{[0,1)}$, which is the rational subset $\{|p| \leq |T| \neq 0\}$. Then

$$\mathcal{O}_X(\mathcal{Y}_{[0,1)}) = \mathcal{O}_X^+(\mathcal{Y}_{[0,1)})[\tfrac{1}{T}],$$

where

$$\mathcal{O}_X^+(\mathcal{Y}_{[0,1)}) = T\text{-adic completion of } \mathbb{Z}_p[\![T]\!][\tfrac{p}{T}].$$

Note that $\mathcal{O}_X(\mathcal{Y}_{[0,1)})$ is Tate, with topologically nilpotent unit T; but it does not contain any non-arch. field!

Explicitly, $\mathcal{O}_X^+(\mathcal{Y}_{[0,1)}) = \underset{(p,T)\text{-adic topology}}{\mathbb{Z}_p[\![T]\!]} + \underset{p\text{-adic topology}}{\mathbb{Z}_p\langle \tfrac{p}{T}\rangle}$, i.e. it consists of $\sum_{n\in\mathbb{Z}} a_n T^n$ s.t.:

- $a_n \in \mathbb{Z}_p \; \forall n$,
- $v_p(a_n) \geq |n| \; \forall n < 0$,
- $v_p(a_n) - |n| \longrightarrow \infty$ as $n \longrightarrow -\infty$.

Further remarks by David:

- Such rings naturally arise in the theory of (φ,Γ)-modules. The ring $\mathcal{O}(\mathcal{Y}_{[0,\frac{1}{r}]})$ is $A_r^{\dagger}$ in Colmez-Cherbonnier.
- The idea that the complement in X of a smaller open disc (e.g. our $\mathcal{Y}_{[0,r)}$) is somehow a "nbhd of a special fibre" is powerful. It explains several previously mysterious properties of modular forms ("spectral halo" of Andreatta-Iovita-Pilloni), where behaviour becomes increasingly simpler and more regular as you go towards the boundary of the unit disc.

Topology: Valuation Spectra

Let A commutative ring. Define the valuation spectrum of A to be

$$\mathrm{Spv}(A) = \{\text{valuations on } A\}/\sim,$$

with topology generated by the subsets

$$\mathcal{U}(f/g) := \{v \in \mathrm{Spv}(A) : v(f) \leq v(g) \neq 0\}.$$

Theorem: (Huber). $\mathrm{Spv}(A)$ is spectral.

Note: For a Huber pair (A,A^+), $\mathrm{Spa}(A,A^+) \subset \mathrm{Spv}(A)$ with the subspace topology.

Question: Can we describe the (equivalence class of) $v: A \to \Gamma \cup \{0\}$ without reference to Γ?

Example: (Riemann-Zariski space). Let $A = K$ be a field. It is a basic fact of commutative algebra that

$$\mathrm{Spv}(K) = \{\text{valuation subrings } R \subset K\}.$$

Recall that this means R is an integral domain with $\mathrm{Frac}(R) = K$ s.t. $\forall x \in K^{\times}$, we have x and/or $x^{-1} \in R$.

Definition: The support of a valuation $v: A \to \Gamma \cup \{0\}$ is

$$\mathfrak{p}_v := v^{-1}(0).$$

It is easy to check:

- $\mathfrak{p}_v$ is prime,
- v induces a valuation $\tilde{v}$ on the "residue field"
$$\kappa(\mathfrak{p}_v) := \mathrm{Frac}(A/\mathfrak{p}_v),$$
- For any given prime $\mathfrak{p} \subset A$ and a valuation $\tilde{v}: \kappa(\mathfrak{p}) \to \Gamma \cup \{0\}$, the composition
$$A \longrightarrow A/\mathfrak{p} \xrightarrow{\tilde{v}} \Gamma \cup \{0\}$$
is a valuation on A.

Proposition:
$$\begin{aligned}\mathrm{Spv}(A) &= \{(\mathfrak{p}, \tilde{v}) : \mathfrak{p} \in \mathrm{Spec}(A),\ \tilde{v} \text{ valuation on } \kappa(\mathfrak{p})\} \\ &= \{(\mathfrak{p}, R) : \mathfrak{p} \in \mathrm{Spec}(A),\ R \subset \kappa(\mathfrak{p}) \text{ valuation subring}\}.\end{aligned}$$

A neat way of organising the study of the topology of $Spv(A)$ is via the natural map
$$\varphi : Spv(A) \longrightarrow Spec(A),$$
sending $v \longmapsto \mathfrak{p}_v$. We think of $Spv(A)$ as a fibration over $Spec(A)$, with fibres $\varphi^{-1}(\mathfrak{p}) = Spv(\kappa(\mathfrak{p}))$.

Proposition: $\varphi : Spv(A) \longrightarrow Spec(A)$ is cts.

Pf: Let $D(a) \subset Spec(A)$ distinguished open. Then
$$\begin{aligned} \varphi^{-1}(D(a)) &= \{ v \in Spv(A) : a \notin \mathfrak{p}_v \} \\ &= \{ v \in Spv(A) : v(a) \neq 0 \} \\ &= \{ v \in Spv(A) : v(0) \leq v(a) \neq 0 \} \\ &= \mathcal{U}(0/a). \end{aligned}$$
$\square$

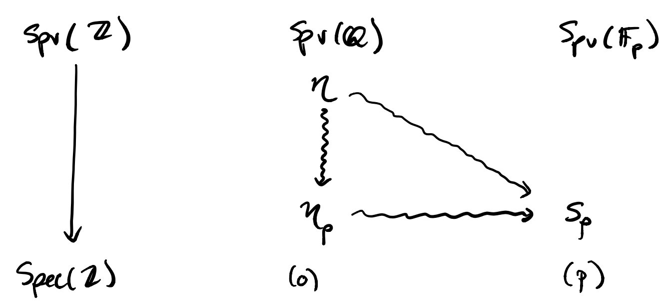
Note there is a natural cts section $Spec(A) \to Spv(A)$, sending $\mathfrak{p} \mapsto (\mathfrak{p}, \text{triv. val.})$.

Example: $Spv(A)$. Recall that $Spv(\mathbb{Z}) = Spa(\mathbb{Z}, \mathbb{Z})$ consists of:

- η, sending $\mathbb{Z} \setminus 0$ to 1;
- for each prime p, $s_p : \mathbb{Z} \to \mathbb{F}_p \to \{0, 1\}$, sending $\mathbb{F}_p^\times \mapsto 1$;
- for each p, $\eta_p : \mathbb{Z} \to \mathbb{Z}_p \to p^{\mathbb{Z}} \cup \{0\}$, p-adic absolute value.

These are:
$$\begin{aligned} \eta &= (0, \text{triv. on } \mathbb{Q}) &&= ((0), \mathbb{Q}) \\ \eta_p &= (0, p\text{-adic a.v. on } \mathbb{Q}) &&= ((0), \mathbb{Z}_{(p)}) \\ s_p &= (p, \text{trivial valuation on } \mathbb{F}_p) &&= ((p), \mathbb{F}_p). \end{aligned}$$

Picture:



Here $\rightsquigarrow$ means specialisation:
$$\overline{\{\eta\}} = Spv(\mathbb{Z}), \qquad \overline{\{\eta_p\}} = \{\eta_p, s_p\}, \qquad \overline{\{s_p\}} = \{s_p\}.$$

We use this picture to understand specialisations in general. Two types:

1. Vertical specialisations: within a common fibre $\varphi^{-1}(\mathfrak{p})$.
2. Horizontal specialisations: moving to a different fibre (in the simplest possible manner).

Lecture Seven: The Closed Unit Disc & The Role of A^+

Aims: 1) picture of $\mathbb{D}_K(0,1) := \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$ ~ closed unit disc
2) explain the role of A^+.

Eg: Let $K = \widehat{\bar{K}}$ non-archimedean, K° = power-bounded elements, $K^{\circ\circ}$ = top. nilpotent elts, $k = K^\circ / K^{\circ\circ}$. Let:

$$D(a,r) = \{x \in K : |x-a| \le r\}, \qquad D(a,r^-) = \{x \in K : |x-a| < r\}.$$

Several flavours of points:

Type I: $a \in K^\circ$, $v_a(f) := |f(a)| \in \mathbb{R}_{\geq 0}$.
$v_a : K\langle T\rangle \longrightarrow \mathbb{R}_{\geq 0}$, "classical/rigid points".

In Pak-Hin's terminology: support of v_a = kernel = $(T-a)$.

Recall: $\{$specialisations of $v \in \mathrm{Spa}(A, A^+)\} \longleftrightarrow \{$valuation rings $\mathrm{im}(A^+) \subseteq R \subseteq \mathcal{O}(v)\}$. value gp

Here:
$$K(v_a) = K, \quad \mathcal{O}(v_a) = K^\circ, \quad \mathrm{im}(A^+) = K^\circ.$$

Type II/III: Let $a \in K^\circ$, $r \in (0,1]$. Let
$$v_{a,r}(f) := \sup_{x \in D(a,r)} |f(x)|, \qquad v_{a,r} : K\langle T\rangle \longrightarrow \mathbb{R}_{\geq 0}.$$

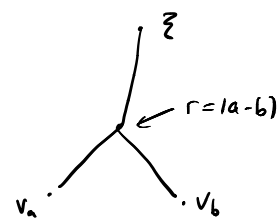
Note again it is non-trivial that this is multiplicative!

Have $\mathrm{Supp}(v_{a,r}) = (0)$.

e.g. $v_{a,1} = v_{0,1} \; \forall a$, $v(\sum a_i T^i) = \sup_i |a_i| = \underline{\text{Gauss point}} =: \xi$.

Fix a, vary r: get
$$\ell_a : [0,1] \longrightarrow \mathbb{D}_K(0,1)$$
$$\ell_a(0) = v_a, \qquad \ell_a(1) = \xi$$

Take $b \in K^\circ$: then
$$\ell_a(r) = \ell_b(r) \iff r \geq |a-b|$$



Def'n: If $r \in |K^\times|$, say $v_{a,r}$ is type II.
(these are <u>branching points</u>).
If $r \notin |K^\times|$, say $v_{a,r}$ is type III.
(non-branching points).

How many branches are there? If $b \neq a$, then ℓ_b meets ℓ_a at $v_{a,r} \iff |b-a| = r$; and $\ell_{b'}$ meets ℓ_b before $v_{a,r} \iff |b-b'| < r$.

... have a "$\mathbb{P}^1$ at every branching point": branches out of $v_{a,r}$ are: • $\mathbb{P}^1(k)$, if $r \neq 1$;
• $\mathbb{A}^1(k)$, $r = 1$.

Type IV: $\mathcal{D} = D_1 \supseteq D_2 \supseteq \cdots$ nested family of discs in K° s.t. $\bigcap_n D_n = \emptyset$. (Cannot happen if K is spherically complete). Then

$$v_{\mathcal{D}}(f) := \inf_n \sup_{x \in D_n} |f(x)|.$$

↪ "dead ends of the tree".

WLOG take D_n radius in $|K^\times|$ (can always change by cofinal family).

↳ so far: Berkovich

Type V: $a \in K^\circ$, $r \in (0,1] \cap |K^\times|$, $? \in \{<, >\}$, $? \neq >$ if $r=1$.

Let $\Gamma_r^? := \{0\} \cup \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, $\quad r' < \gamma < r \;\; \forall r' < r$ if $? = <$,
$\quad r < \gamma < r' \;\; \forall r' > r$ if $? = >$.

Have

$$v_{a,r}^? : K\langle T\rangle \longrightarrow \{0\} \cup \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}},$$

$$f = \sum a_i (T-a)^i \longmapsto \sup |a_i| \gamma^i$$

with support $= 0$.

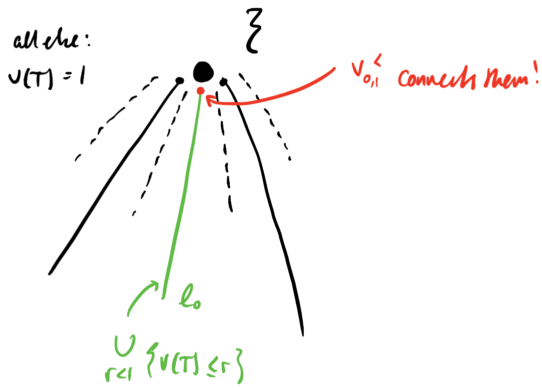
Have $v_{a,r}^<(f) \leq v_{a,r}^<(g) \iff$ either $\sup_i |a_i| r^i < \sup_i |b_i| r^i$,
or $\sup_i |a_i| r^i = \sup_i |b_i| r^i$ and the a_i sup occurs no earlier than the b_i sup.

Thus: If $v_{a,r}^<(f) \leq v_{a,r}^<(g) \implies v_{a,r}(f) \leq v_{a,r}(g)$.
But the converse does not hold!

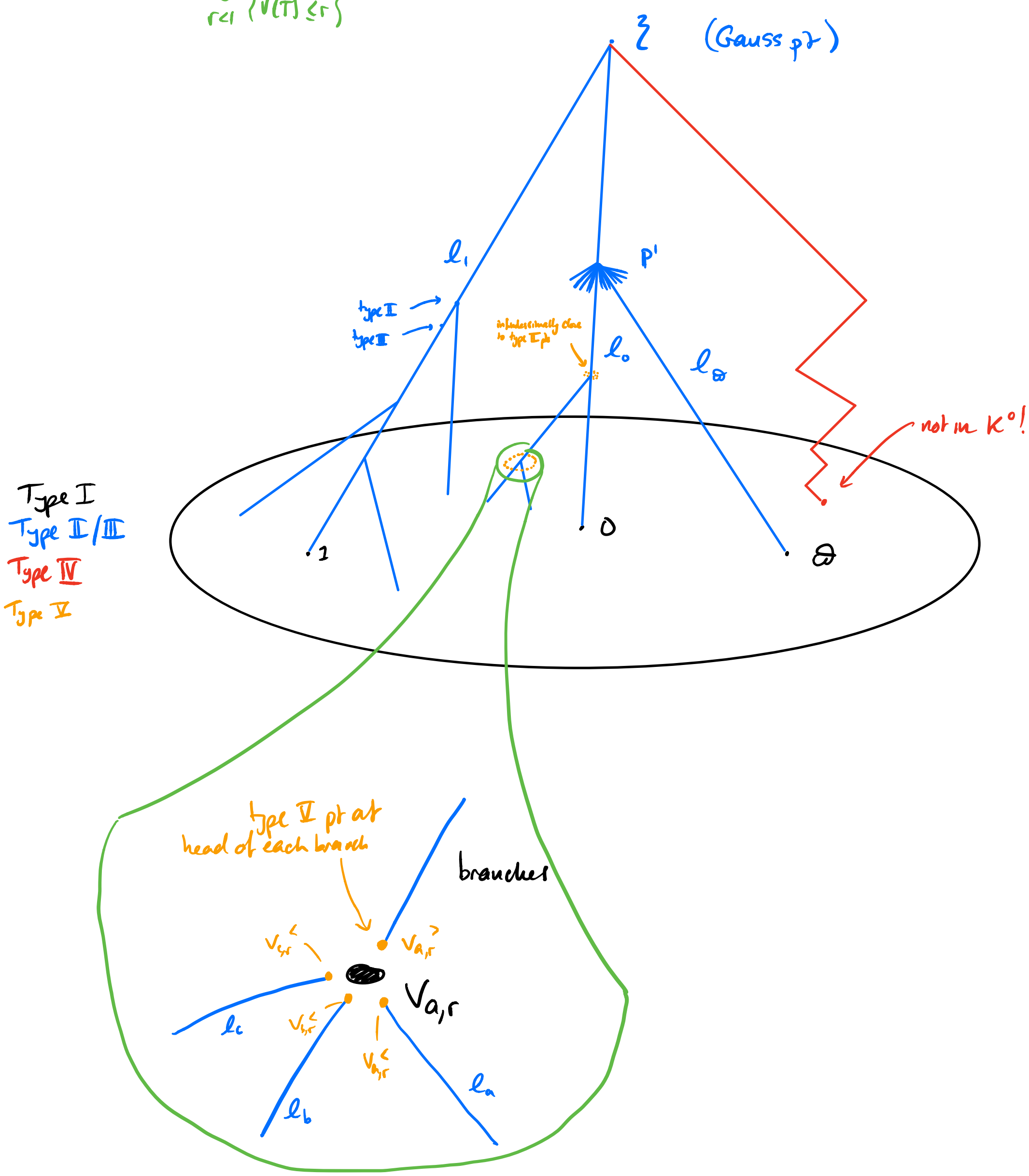
↳ Thus: $v_{a,r}^<$ is a proper specialisation of $v_{a,r}$.

Check: this is everything. $K[T] \in K\langle T\rangle$
$\rightsquigarrow v$ det. by $v(T-a)$.

Type I, III, IV, V closed, II not closed.



all else:
v(T) = 1
ξ
$v_{0,1}^{<}$ connects them!
ℓ_0
$\bigcup_{r<1} \{v(T) \le r\}$



ξ (Gauss pt)
ℓ_1
p'
type I
type II
infinitesimally close to type II pt
ℓ_0
ℓ_∞
not in K°!
Type I
Type II/III
Type IV
Type V
1
0
∞
type V pt at head of each branch
branches
$v_{c,r}^{<}$
$v_{a,r}^{>}$
$V_{a,r}$
ℓ_c
$v_{b,r}^{<}$
$v_{a,r}^{<}$
ℓ_b
ℓ_a

§2: The role of A^+.

Let $K = \mathbb{C}_p$, so $\mathrm{Spv}(K) = \mathrm{Spa}(K, K^\circ) = \{|\cdot|\}$.

What is $\mathrm{Spv}(\mathbb{C}_p\langle T\rangle)$?
↑ all cts valuations.

- $v \in \mathrm{Spv}(\mathbb{C}_p\langle T\rangle) \Rightarrow v|_{\mathbb{C}_p}$ cts, so $v|_{\mathbb{C}_p} \sim |\cdot|$, $\quad v(\mathcal{O}_{\mathbb{C}_p}) \leq 1$.

So $\mathrm{Spv}(\mathbb{C}_p\langle T\rangle) =$ "$\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p})$"
↑ not strictly valid!

- $v \in \mathrm{Spv}(A) \Rightarrow \{a \in A : v(a) < 1\}$ is an open prime ideal:
$v \in \mathrm{Spv}(\mathbb{C}_p\langle T\rangle) \Rightarrow v(\mathfrak{m}_{\mathbb{C}_p}\langle T\rangle) < 1$.

So:

$$\mathrm{Spv}(\mathbb{C}_p\langle T\rangle) = \mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p} + \mathfrak{m}_{\mathbb{C}_p}\langle T\rangle).$$

↑ this is now allowed: it is the "smallest allowed integral ring"

"Spv is always a Spa really!"

What is $\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p} + \mathfrak{m}_{\mathbb{C}_p}\langle T\rangle)$? What else have we picked up?

$$(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle) \geq (\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p} + \mathfrak{m}_{\mathbb{C}_p}\langle T\rangle) \geq (\mathbb{C}_p\langle pT\rangle, \mathcal{O}_{\mathbb{C}_p}\langle pT\rangle).$$

$$\mathbb{D}_{\mathbb{C}_p}(0,1) \subseteq \mathrm{Spv}(\mathbb{C}_p\langle T\rangle) \subseteq \mathbb{D}_{\mathbb{C}_p}(0, |p^{-1}|).$$

If $v \in \mathbb{D}_{\mathbb{C}_p}(0, |p^{-1}|)$ satisfies $v(T) \geq r$, some $r \in \mathbb{R}$, $r > 1$, then $v \notin \mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p} + \mathfrak{m}_{\mathbb{C}_p}\langle T\rangle)$.
(pick $x \in \mathfrak{m}_{\mathbb{C}_p}$, $r^{-1} < |x| < 1$.

"extended closed unit disc"

... upshot:

$$\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p} + \mathfrak{m}_{\mathbb{C}_p}\langle T\rangle) \setminus \mathbb{D}_{\mathbb{C}_p}(0,1) = \{v_{0,1}^{>}\}$$

↑ the missing point!

... we've recovered a point we were missing, by changing A^+.

Lemma: $\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p} + \mathfrak{m}_{\mathbb{C}_p}\langle T\rangle)$ is <u>proper</u> over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$.

"proof": $\mathbb{D}_{\mathbb{C}_p}(0,1) \hookrightarrow \mathbb{D}_{\mathbb{C}_p}(0, |p^{-1}|) \hookrightarrow \mathbb{P}^{1,an}_{\mathbb{C}_p}$; then $\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p} + \mathfrak{m}_{\mathbb{C}_p}\langle T\rangle)$ is the closure of $\mathbb{D}_{\mathbb{C}_p}(0,1)$ inside $\mathbb{P}^{1,an}_{\mathbb{C}_p}$. (□)

↳ this gives us a "canonical compactification" of $\mathbb{D}_{\mathbb{C}_p}(0,1)$.

... and this works in greater generality!

Let $A/\mathbb{C}_p$ affinoid algebra; then $\mathrm{Spa}(A, \mathcal{O}_{\mathbb{C}_p} + A^{\circ\circ})$ is a canonical compactification of $\mathrm{Spa}(A, A^\circ)$.

"adic spaces have canonical partial compactifications".
↑ it's not always proper! It <u>is</u> proper if $\mathrm{Spa}(A, A^\circ)$ is quasi-compact.

e.g. open unit disc is its own cpt. compact.

Rmk: This is why Huber introduced this: allows cptly supported étale cohomology of rigid spaces. (Serre duality etc).

(David: analogies to Grosse-Klönne dagger spaces: cpt supported de Rham & coherent cohomology).

Lecture Eight: Perfectoid Fields

Nadav Gropper

Fontaine–Wintenberger, 1970s: gave a correspondence between the Galois groups of $\underbrace{\widehat{\mathbb{Q}_p(p^{1/p^\infty})}}_{``K}$ and $\underbrace{\widehat{\mathbb{F}_p((t))(t^{1/p^\infty})}}_{``K^\flat}$, and in particular an equivalence of categories:

$$\{\text{finite extensions } L \text{ of } K\} \xrightarrow{\sim} \{\text{finite extensions } L^\flat \text{ of } K^\flat\}.$$

A concrete example: (cf. MO post) if $L^\flat$ is the field given by $X^2 - 7tX + t^5$; want to take L via $t \leftrightarrow p$, $X^2 - 7pX + p^5$. However if e.g. $p = 3$ then

$$X^2 - 7tX + t^5 = X^2 - tX + t^5,$$

but we don't expect the fields cut out by $X^2 - 7pX + p^5$ and $X^2 - pX + p^5$ to be the same: hence need to add the p-power roots to rectify.

Def'n: A field K is called perfectoid if it's complete, non-discretely valued, with perfect residue field char. $(0,p)$ or (p,p), and $x \longmapsto x^p$ is surjective on $\mathcal{O}_K/p$.

Remark: every element of $|K^\times|$ is a p^{th} power.

Examples:
1) $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$,
2) $\widehat{\mathbb{F}_p((t))(t^{1/p^\infty})}$,
3) $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$,
4) $\mathbb{C}_p$.

Non-example: $\widehat{\bigcup_{p \nmid n} \mathbb{Q}_p(\mu_n)}$ (completion of maximal unramified extension).
$\rightsquigarrow$ not surjective on $\mathcal{O}_K/p$.

Have the tilting procedure allowing transfer from mixed char $(0,p)$ to (p,p). If K perfectoid, the tilt of K is

$$K^\flat = \varprojlim_{x \mapsto x^p} K,$$

with addition

$$(x^{(0)}, x^{(1)}, \ldots) + (y^{(0)}, y^{(1)}, \ldots) = (z^{(0)}, z^{(1)}, \ldots),$$

where

$$z^{(i)} = \lim_{n \to \infty} \left(x^{(i+n)} + y^{(i+n)}\right)^{1/p^n}.$$

This comes from $\mathcal{O}_K^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_K \cong \varprojlim_{\Phi} \mathcal{O}_K/p$ (using: $x \equiv y \pmod{p^n} \Rightarrow x^p \equiv y^p \pmod{p^{n+1}}$).
(Φ Frobenius)

Then $K^\flat = \mathrm{Frac}(\mathcal{O}_K^\flat)$.

Given a sequence $(x^{(0)}, x^{(1)}, \ldots)$, we have a multiplicative map

$$\begin{array}{ccccc} K^\flat & = & \varprojlim K & \longrightarrow & K \\ f & & \longmapsto & & f^\# \end{array}$$

by projection onto the zeroth co-ordinate.

Lemma: The map
$$|\cdot|_b : \mathcal{O}_{K^\flat} \ni \alpha \longmapsto |\alpha^\#|$$
defines a non-arch. abs. value. We have:

1) $|\mathcal{O}_{K^\flat}|_b = |\mathcal{O}_K|$ (same value groups),
2)

Proposition: K^b with $|\cdot|_b$ is a perfect complete non-arch field of char p, $\mathcal{O}_{K^b} = \varprojlim \mathcal{O}_K/p^n\mathcal{O}_K$. (?)

Facts: (1) $\mathbb{C}_p^{\,b}$ is algebraically closed.

(2) Finite extensions of perfectoid fields are perfectoid.

(3) Suppose
$$\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^{\,b} \subset F \subset \mathbb{C}_p^{\,b};$$
then want $F^\#$ s.t. $(F^\#)^b = F$.

$\hookrightarrow$ several ways to do this: Witt vectors, almost mathematics.

If such $F^\#$ exists, then F alg. closed $\rightsquigarrow$ $F^\#$ alg. closed. (in eg. above, this forces $F = \mathbb{C}_p^{\,b}$).

Perfectoid rings

Definition: A complete Tate p-ring R is called perfectoid if R is uniform, the pseudo-uniformizer $\varpi \in R$ s.t. ϖ^p / p in R°, and s.t.
$$\Phi : R^\circ/\varpi \longrightarrow R^\circ/\varpi^p$$
is an isomorphism.

Again, have $R^b = \varprojlim_{x \mapsto x^p} R$, and untilting.

Theorem: A Huber pair (R, R^+), with R perfectoid, is sheafy.

Let $X = \mathrm{Spa}(R, R^+)$. Define $X^b = \mathrm{Spa}(R^b, (R^+)^b)$. There is a homeomorphism $X \longrightarrow X^b$ which preserves rational subsets. If $\mathcal{U} \subseteq X$ rational subset, then $\mathcal{O}_X(\mathcal{U})$ is perfectoid.

Definition: A perfectoid space is an adic space which is covered by adic affinoids $\mathrm{Spa}(R, R^+)$ with R perfectoid.

The tilting procedure glues to a functor on perfectoid spaces, and tilting is an equivalence of categories.

Also have a good notion of étale sites on perfectoid spaces; and tilting gives an equivalence of sites.

Why perfectoids?

$L = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, $L^b = \widehat{\mathbb{F}_p((t))(t^{1/p^\infty})}$. Have $|\mathbb{A}_{K^b}^{1,\mathrm{ad}}| \cong \varprojlim |\mathbb{A}_K^{1,\mathrm{ad}}|$, hence projection
$$\mathrm{pr} : \mathbb{P}^n_{L^b} \longrightarrow \mathbb{P}^n_L \supset X \text{ smooth complete intersection.}$$

Note $\mathrm{pr}^{-1}(X)$ will not be given by equations (pr is transcendental), but we have a Galois-equivariant injective map
$$H^i(X) \longrightarrow H^i(\mathrm{pr}^{-1}(X))$$

on ℓ-adic cohomology theories.

<u>Lemma:</u> Let $\tilde{X}$ small open neighbourhood of X, then there is a hypersurface $Y \subset pr^{-1}(\tilde{X})$ ("approximation algorithm")
... pass through Y to get to desired result.

<u>LECTURE NINE: THE PERFECTOID MODULAR CURVE</u>

David Loeffler

<u>§1: Setting</u>

Let $n \geq 3$. Then $\exists$ an alg. variety $Y(n)/\mathbb{Q}$ (full lvl modular curve) s.t.

$$Y(n)(\mathbb{C}) \cong \Gamma(n)\backslash\mathcal{H} \times (\mathbb{Z}/n\mathbb{Z})^\times,$$

and for any char 0 field,

$$Y(n)(L) = \{(E,P,Q) : E/L \text{ elliptic curve}, P,Q \text{ basis of } E[n]\}$$

Make it proper: $X(n) = Y(n) \cup \{\text{cusps}\}$, smooth proper.

<u>Problem:</u> For fixed p, understand the tower $X(p) \leftarrow X(p^2) \leftarrow \cdots$

(Important, but pretty hard: reduction gets <u>increasingly</u> bad!)

Want to make sense of inverse limits of adic spaces. Not obvious: limits of Huber rings might not be Huber!

<u>Def'n:</u> Let

$$\begin{tikzcd} X_1 & X_2 \arrow[l] & \cdots \arrow[l] \\ & & X \arrow[ull, "f_1"] \arrow[ul, "f_2"'] \end{tikzcd}$$

inverse system of adic spaces, X adic mapping to each X_i.

Say "$X \sim \varprojlim X_i$" tilde-limit if:

a) $|X| \longrightarrow \varprojlim_i |X_i|$ isom. of top. spaces,

b) X has a covering $X = \cup\, \mathcal{U}_j$ by affinoids, s.t.

$$\varinjlim_i \mathcal{O}_{X_i}(f_i^{-1}(\mathcal{U}_j)) \longrightarrow \mathcal{O}_X(\mathcal{U}_j)$$

has <u>dense image</u> $\forall j$.

<u>Thm:</u> (Scholze). If X_i adic over $\mathrm{Spa}(C, \mathcal{O}_C)$, C perfectoid field, X perfectoid, then it is <u>unique</u>.

($\exists$ counterexample in general in Scholze-Weinstein: constant system on adic X with two <u>different</u> topologies).
$\rightsquigarrow$ not possible to get such perfectoid examples!

First arose from p-divisible groups - these are a toy example.

<u>§3: Perfectoid modular curve</u>

<u>Theorem:</u> (Scholze). $\exists$ a perfectoid space $\mathcal{X}/\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ s.t. $\mathcal{X} \sim \varprojlim \mathcal{X}(p^n)^{ad}_{\mathbb{C}_p}$. (or: any perfectoid extn of $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$).

likely need to contain this: structure of finite lvl modular curves require it

Basic outline of proof: break tower into steps, $\Gamma_0(p^n)$, $\Gamma_1(p^n)$, $\Gamma(p^n)$.
Strategy: first look at

$$\cdots \longrightarrow \mathcal{X}_0(p^n) \longrightarrow \mathcal{X}_0(p^{n-1}) \longrightarrow \cdots,$$

classifying elliptic curves with a cyclic subgp C order p^n.

$\mathcal{X}_0(p^n)_a$ = "anticanonical locus" = locus where E is <u>ordinary</u>, and C is complementary to $\hat{E}[p^n]$ (torsion in formal group). (Also have <u>canonical</u> locus, $C = \hat{E}[p^n]$).
↳ Igusa tower: more common!)

$\exists$ a formal scheme model s.t.

$$\mathcal{X}_0(p^n)_a \longrightarrow \mathcal{X}_0(p^{n-1})_a$$

<u>is Frobenius mod p</u>:
↳ can build $\mathcal{X}_0(p^\infty)_a$ as an adic space, and it's perfectoid.

- "Overconvergence": for $\varepsilon < 1/2$, can make sense of $\mathcal{X}_0(p^\infty)(\varepsilon)_a$: "things less than ε away" from $\mathcal{X}_0(p^\infty)_a$, + argument extends
($\rightsquigarrow$ valuation of Hasse-invariant $< \varepsilon$).

- Extend from $\Gamma_0(p^\infty)$ to $\Gamma_1(p^\infty)$, then $\Gamma(p^\infty)$: headaches at the cusps. (Maps are étale away from cusps).
↳ Shimura curves: cf. Chojecki-Hansen-Johansson. (Ben Heuer's thesis?)

- Use action of $GL_2(\mathbb{Q}_p)$ to extend from anti-can. locus: $\mathcal{X}(p^n)$ is covered by finitely many translates of $\mathcal{X}(p^n)(\varepsilon)_a$.
$\rightsquigarrow$ perfectoid by transport of structure. (□)

<u>§4: The period map</u>
Theorem is beautiful, but what is it good for?

↳ most useful because of a "by-product" of construction.

<u>Theorem</u>: $\exists$ morphism of adic spaces (Hodge-Tate period map)

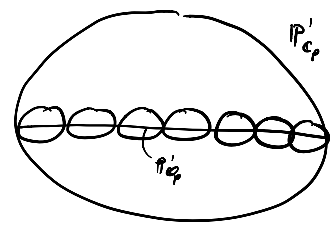
$$\pi_{HT} : \mathcal{X}(p^\infty) \longrightarrow (\mathbb{P}^1_{\mathbb{C}_p})^{ad}.$$

Idea: $\mathbb{C}_p$-pts of $\mathcal{X}(p^\infty)$ (non-cuspidal) $\longleftrightarrow$ ell. curves $E/\mathbb{C}_p$ + basis of $T_p(E) \simeq \mathbb{Z}_p^2$.

Hodge-Tate decomposition: $T_p(E) \otimes \mathbb{C}_p \hookleftarrow \Omega^1_E \otimes \mathbb{C}_p$ canonical 1-dim subspace.

↳ line inside 2-dim space: element of $\mathbb{P}^1$!

Quite strange morphism: <u>all</u> of ordinary locus sent to $\mathbb{P}^1_{\mathbb{Q}_p}$! so non-ord. locus sent to p-adic $\mathcal{H}_p$



Crucial application: pullbacks of affinoids in $\mathbb{P}^1$ give affinoid covers of $\mathcal{X}(p^\infty)$. (useful for studying cohomology via Čech complexes).

$\longrightarrow$ his main application: compute cohomology (fake Hasse invariants from π_{HT}; chop $\mathcal{X}(p^\infty)$ into palatable pieces).

Lecture Ten: Perfectoid Modular Forms

Chris Williams

[Caveat: huge topic, hard to know what to leave out]

§1: Classical modular forms

$\Gamma = \Gamma_1(N) \leq SL_2(\mathbb{Z})$, k positive integer. equivalent ways of defining $M_k(\Gamma)$:

(ANALYTIC) $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. M.f. is

$$f : \mathcal{H}^* \longrightarrow \mathbb{C} \quad \text{holomorphic,}$$

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \qquad \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

($\mathbb{C}$ - GEOMETRIC) Let $\mathrm{pr} : \mathcal{H}^* \longrightarrow \Gamma \backslash \mathcal{H}^* =: X_\Gamma$. Define sheaf ω_k on X_Γ by

$$\omega_k(V) := \{ f : \mathrm{pr}^{-1}(V) \subset \mathcal{H}^* \to \mathbb{C} \text{ holo.} \mid f(\gamma\tau) = (c\tau+d)^k f(\tau) \}$$

$$\rightsquigarrow \quad M_k(\Gamma) = \omega_k(X_\Gamma) = H^0(X_\Gamma, \omega_k).$$

Facts:
- ω_k is a line bundle (loc. rk 1)
- both X_Γ and ω_k admit models over nice rings R

(ALGO - GEOMETRIC) $M_k(\Gamma, R) = H^0(X_{\Gamma/R}, \omega_k)$

↳ have q-expansions in $R[\![q]\!]$.

At heart of these results: moduli interpretation,

$$X_\Gamma(R) \sim \{ E/R \text{ ell. curve} + \text{"}\Gamma\text{-level structure"} \}.$$

(ALGEBRAIC) reinterpret: mod. form of wt k, lvl Γ over R is:

(1) $\quad f : \{ E/R + (\text{extra data}) \} \longrightarrow R$

$\quad$ + "functorial of wt k" in (extra data).

§2: p-adic modular forms à la Serre

Guiding question: p prime, $f \in M_k(\Gamma_1(N), \overline{\mathbb{Q}})$. Let $k_m := k + p^m \longrightarrow k$ p-adically.

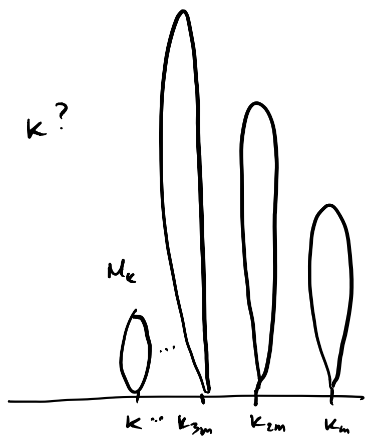
Q: Do $\exists$ Hecke eigenforms $f_m \in M_{k_m}(\Gamma_1(N), \overline{\mathbb{Q}})$ s.t. $f_m \to f$ as $m \to \infty$?

ie. $f(\tau) = \sum a_n q^n$, $f_m(\tau) = \sum a_n^m q^n$, and $a_n^m \to a_n$ $\forall n$?
(uniformly)

Naive reinterpretation: (systematic approach)

Q: Can I p-adically deform $M_k(\Gamma)$ as I deform k?

A: No, for dimension reasons:



$\longrightarrow$ need to work with larger (∞-dim.) spaces.

First definition (Serre):

$$M_k^{p\text{-adic}}(\Gamma(1), \mathbb{Z}_p) = \left\{ f(q) \in \mathbb{Z}_p[\![q]\!] : \exists f_i \in M_{k_i}(\Gamma(1), \mathbb{Z}) \text{ s.t. } f_i \to f,\ k_i \to k \right\}$$

= "p-adic completion of mod forms"

Remarks: 1) already very useful for studying congruences between m.f. (Ramanujan)

2) ... but this space is <u>too big</u>!

e.g. $\lambda \in p\mathbb{Z}_p$, $f \in M^{p\text{-adic}}$, $f_\lambda := (1 - V_p \lambda)^{-1}(1 - V_p U_p) f \in M^{p\text{-adic}}$

Then $U_p f_\lambda = \lambda f_\lambda$

$\rightsquigarrow$ $\exists$ pathological eigenforms; spectrum of U_p is continuous

$\rightsquigarrow$ no good spectral theory of Hecke operators!

(Can't expect $M^{p\text{-adic}}$ to tell us anything about classical eigenforms).

§3: Overconvergent modular forms

Coleman: geometric fix. Let $X = X_{SL_2(\mathbb{Z})}$.

Fact: $\exists$ modular form A over $\mathbb{Z}_p$ s.t.:

(1) $A(E, \text{data}) \in \begin{cases} \mathbb{Z}_p^\times & : E \text{ ordinary at } p, \\ p\mathbb{Z}_p & : E \text{ supersingular at } p, \end{cases}$

(2) $A(q) \equiv 1 \pmod p$ in $\mathbb{Z}_p[\![q]\!]$.

(If $p \geq 5$, can take $A = E_{p-1}$ Eisenstein).

Lemma: A is invertible in $M^{p\text{-adic}}(\Gamma(1), \mathbb{Z}_p)$.

Pf: $A^{p^n}(q) \equiv 1 \pmod{p^n}$

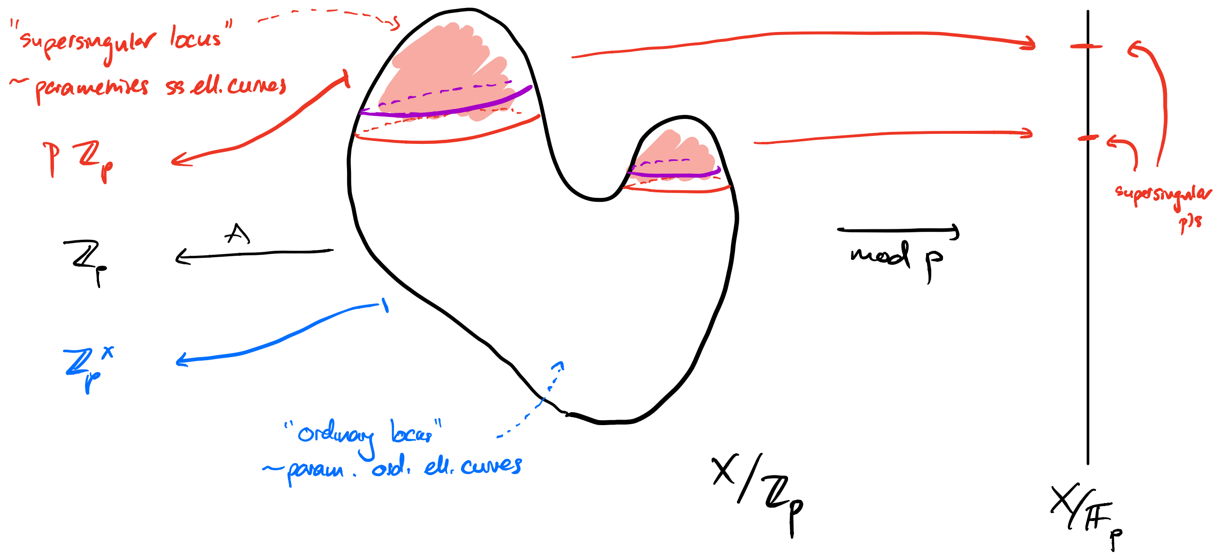
$\rightsquigarrow \lim_{n\to\infty} A^{p^n}(q) = 1$

$\rightsquigarrow \lim_{n\to\infty} A^{p^n-1}(q) = 1/A.$ □

Observe: E ss $\rightsquigarrow A(E, \text{data}) \in p\mathbb{Z}_p$ not invertible.

$\rightsquigarrow 1/A$ only well-defined on (E, data) with E ordinary.

Hence: want to make sense of "ordinary locus" $X^{\text{ord}} \subset X$.



$\rightsquigarrow$ X^{ord} = subspace where $|A| = 1$. (meaningless in Zariski... but definable in rigid world!)

<u>Def'n:</u> $(X/\mathbb{Z}_p, \omega_k) \xrightsquigarrow{\text{GAGA}} (\mathfrak{X}/\mathrm{Spf}\,\mathbb{Z}_p, \omega_k)$ formal scheme

Let $\mathfrak{X}^{ord} := \mathfrak{X}(|A| = 1) \subseteq \mathfrak{X}$.

<u>Thm:</u> $M_k^{p\text{-adic}}(SL_2(\mathbb{Z}), \mathbb{Z}_p) \cong H^0(\mathfrak{X}^{ord}/\mathrm{Spf}\,\mathbb{Z}_p, \omega_k)$.

Have:

$$M_k = H^0(\mathfrak{X}, \omega_k) \quad \text{(too small)},$$
$$\cap \qquad\qquad \cap$$
$$M_k^{p\text{-adic}} = H^0(\mathfrak{X}^{ord}, \omega_k) \quad \text{(too big)}.$$

Def'n: (Coleman). consider $0 \leq \varepsilon < \frac{p}{p+1}$

$$\mathfrak{X}^{ord} \subset \mathfrak{X}[\varepsilon] \subset \mathfrak{X}$$
$$\mathfrak{X}[\varepsilon] = \mathfrak{X}(|A| \geq |p^{\varepsilon}|).$$

~ parametrises ell. curves that are ordinary or "not <u>too</u> supersingular".

Define ε-<u>overconvergent modular forms of wt k</u> to be

$$M_k \subset M_k^{\dagger} := H^0(\mathfrak{X}[\varepsilon], \omega_k) \subset M_k^{p\text{-adic}}.$$

Facts: 1) $M_k^{\dagger}$ has discrete spectrum of Hecke eigenvalues
$\longrightarrow$ not too big now!

2) after adding level $\Gamma_0(p)$-structure, spaces $M_k^{\dagger}$ vary p-adically in k.
$\longrightarrow$ <u>do</u> get eigenforms $f_m \to f$, for <u>any</u> f!

§4: Analytic o.c. mod. forms

	<u>Analytic</u>	Geometric
Classical	$f : \mathcal{H} \to \mathbb{C}$	$H^0(X, \omega_k)$
Overconvergent		$H^0(\mathfrak{X}[\varepsilon], \omega_k)$
p-adic (Serre)		$H^0(\mathfrak{X}[0], \omega_k)$

Qs: 1) missing analytic def'ns?

2) more general p-adic weights?

$\mathbb{C}$-picture:

co-ord $z := \pi^* \xi$ on $\quad\quad$ co-ord. ξ

$$
\begin{tikzcd}
\text{func. } (cz+d) \text{ on } \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) \arrow[r, "\pi"] \arrow[d, "pr"', "/\Gamma"] & \mathbb{P}^1(\mathbb{C}) \\
\omega_k \arrow[r] & X_\Gamma
\end{tikzcd}
$$

(defined using) $\rightsquigarrow$ ω_k

p-adic:

$z := \pi_{HT}^* \xi$ on $\quad\quad$ co-ord ξ on

$$
\begin{tikzcd}
\text{func } (cz+d) \text{ on } \mathcal{X}_{\Gamma(p^\infty)}[\varepsilon]_a \arrow[r, "\pi_{HT}"] \arrow[d, "pr"', "``/SL_2(\mathbb{Z}_p)\text{''}"] & \mathbb{P}^1 \\
\omega_k \arrow[r] & \mathcal{X}[\varepsilon]
\end{tikzcd}
$$

<u>Thm</u>: (Chojecki-Hansen-Johansson, Birkbeck-Heuer-W.)

An ε-overconvergent modular form is also a perfectoid function

$$f : \mathcal{X}_{\Gamma(p^\infty)}[\varepsilon]_a \longrightarrow \mathbb{C}_p$$

s.t.

$$f(\gamma\tau) = (cz+d)^{-k} f(\tau) \qquad \forall \gamma \in SL_2(\mathbb{Z}_p).$$

<u>Def'n</u>: a p-adic weight is a character

$$\kappa : \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times.$$

e.g. $k \in \mathbb{Z}$, $\quad \kappa(x) = x^k$

$\rightsquigarrow$ κ is a p-adic weight.

<u>Def'n</u>: κ a p-adic wt. Define

$$M_\kappa^\dagger(\Gamma_0(p)) := \Big\{ f : \mathcal{X}_{\Gamma(p^\infty)}[\varepsilon]_a \longrightarrow \mathbb{C}_p : \; f(\gamma\tau) = \kappa^{-1}(cz+d) f(\tau) \;\; \forall \gamma \in \Gamma_0(p) \Big\}.$$