Explicit CFT for Real Quadratic Fields

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§1: Motivation

<u>Recap</u>: CFT completely describes abelian extensions of number fields – <u>theoretically</u>

... but what about <u>explicit</u> results?

We've seen:

- over $\mathbb{Q}$: Kronecker-Weber says: if $L/\mathbb{Q}$ abelian, then $L \subset \mathbb{Q}(\zeta_m)$, some m. (N last)
- $K = \mathbb{Q}(\tau)$ Imaginary quadratic field, [ie. $a\tau^2 + b\tau + c = 0$, $b^2 - 4ac < 0$]
  if L/K abelian, then $L \subset K(j(\tau), h(E_{tors}))$, $E = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.
  (h: Weber function)
- More general fields?...

<u>Weaker analogues</u>:

- Over $\mathbb{Q}$: $\mathbb{Q}(\exp(2\pi i/n))/\mathbb{Q}$ is abelian
- Over $K = \mathbb{Q}(\tau)$ IQF:

$$K(j(\tau))/K \text{ is abelian.}$$

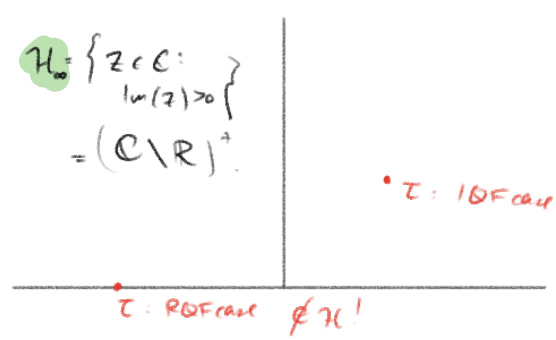
+ K^{ab} gen. by $j(\tau)$, $\wp(\tau, z)$ (torsion pts.)

<u>Hilbert's 12th problem</u>: K a number field. Is there an "analytic function" J s.t. K^{ab} is generated by values of J?

Today: consider $K = \mathbb{Q}(\tau)$ a real quadratic field; so $a\tau^2 + b\tau + c = 0$, $b^2 - 4ac > 0$.

§2: RM points in the "upper half-plane"

$\mathcal{H}_\infty = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\} = (\mathbb{C} \setminus \mathbb{R})^+$



<u>Guess 1</u>: What about $K(j(\tau))$?

<u>Problem</u>: j is a function on $\mathcal{H}_\infty$...

... and if $\tau \in \mathbb{R}$, then $\tau \notin \mathcal{H}_\infty$

Darmon realised a way around. Let $K/\mathbb{Q}$ quadratic, and consider splitting behaviour of the place ∞ for $\mathbb{Q}$.

Let $\iota : K \hookrightarrow \mathbb{C}$ embedding. Then:

K imaginary	K real
$K/\mathbb{Q}$ <u>not split</u> at ∞: $K\otimes\mathbb{R}\cong\mathbb{C}$	$K/\mathbb{Q}$ <u>split</u> at ∞: $K\otimes\mathbb{R}\cong\mathbb{R}\times\mathbb{R}$
$\iota(K)\not\subset\mathbb{R}=\partial\mathcal{H}_\infty$	$\iota(K)\subset\mathbb{R}=\partial\mathcal{H}_\infty$
so $\exists\,\tau\in K$, $\iota(\tau)\in\mathcal{H}_\infty$	so $\not\exists\,\tau\in K$, $\iota(\tau)\in\mathcal{H}_\infty$.

... Darmon: consider other places of $\mathbb{Q}$, ie. <u>p-adic</u> places!

<u>Def'n</u>: Let p prime. The p-adic upper half-plane is

$$\mathcal{H}_p=\mathbb{C}_p\setminus\mathbb{Q}_p,\qquad \mathbb{C}_p:=\text{completion of }\overline{\mathbb{Q}}_p.$$

Now let $K/\mathbb{Q}$ real quadratic. Let $\iota_p: K\hookrightarrow\mathbb{C}_p$ embedding.

K not split at p	K split at p
$K\otimes\mathbb{Q}_p\cong K_p$ (quadratic $/\mathbb{Q}_p$)	$K\otimes\mathbb{Q}_p\cong\mathbb{Q}_p\times\mathbb{Q}_p$
$\iota_p(K)\not\subset\mathbb{Q}_p=\partial\mathcal{H}_p$	$\iota_p(K)\subset\mathbb{Q}_p=\partial\mathcal{H}_p$
so $\exists\,\tau\in K$, $\iota_p(\tau)\in\mathcal{H}_p$	so $\not\exists\,\tau\in K$, $\iota_p(\tau)\in\mathcal{H}$.

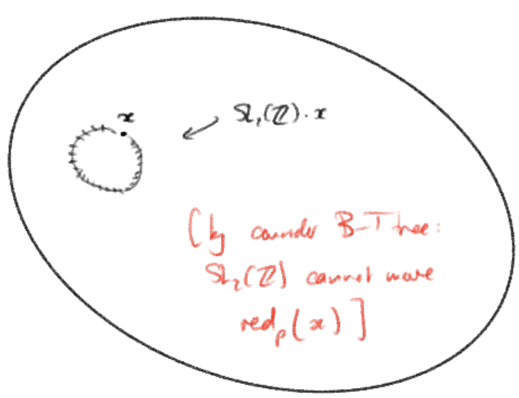
So if p is inert in K, then $\exists$ interesting supply of "RM points" in $\iota_p(K)\cap\mathcal{H}_p$.

§3: Rigid Meromorphic Cocycles

<u>Taking stock</u>:
- K IQF: $j:\mathcal{H}\to\mathbb{C}$ "unique" $SL_2(\mathbb{Z})$-inv. meromorphic function.
- K RQF: does there exist $J:\mathcal{H}_p\to\mathbb{C}$ $SL_2(\mathbb{Z})$-invariant?

Problem: $SL_2(\mathbb{Z})$ is <u>too small</u>; you can never get far from x using $SL_2(\mathbb{Z})$.

↳ "too many $SL_2(\mathbb{Z})$-inv. functions, and can't expect them to be interesting" (I think).



Better: use $\Gamma=SL_2(\mathbb{Z}[\frac{1}{p}])$. So:

Does there exist $J:\mathcal{H}_p\longrightarrow\mathbb{C}_p$ p-adic meromorphic, + Γ-invariant?

... No!! Because $\forall x \in \mathcal{H}_p$, Γx is dense in $\mathcal{H}_p$
$\Rightarrow$ any such J is constant.

Reinterpretation: Let $\mathcal{M}_{\mathbb{C}}^\times :=$ meromorphic fns on $\mathcal{H}_\infty$
$\mathcal{M}^\times =$ p-adic meromorphic functions on $\mathcal{H}_p$.

Then

$$j \in H^0(SL_2(\mathbb{Z}), \mathcal{M}_{\mathbb{C}}^\times)$$

and

$$H^0(\Gamma, \mathcal{M}^\times) = \text{constants} = \mathbb{C}_p^\times.$$

$\boxed{\text{Darmon–Vonk: consider } H^1(\Gamma, \mathcal{M}^\times) \text{ instead!!}}$ (Duke, 2021)

Def'n: 1) Let $\Gamma_\infty \subset \Gamma = SL_2(\mathbb{Z}[\frac{1}{p}])$ be the subgroup of upper-triangular matrices.

2) A rigid meromorphic cocycle is a function

$$J : \Gamma \longrightarrow \mathcal{M}^\times$$

s.t.: 1) [cocycle] $J(\gamma_1 \gamma_2) = J(\gamma_1) \times \gamma_1 J(\gamma_2)$, $\forall \gamma_1, \gamma_2 \in \Gamma$;
2) ["quasi-parabolic"] $J|_{\Gamma_\infty}$ is constant, i.e. $\exists \alpha \in \mathbb{C}_p^\times$ s.t.

$$J\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \alpha \qquad \forall a,b,d.$$

Each J gives a class in $H^1_f(\Gamma, \mathcal{M}^\times) :=$ "quasi-parabolic cohomology classes".

§4: RM values of rigid meromorphic cocycles

Let $J : \Gamma \longrightarrow \mathcal{M}^\times = \{\text{mero fns } \mathcal{H}_p \longrightarrow \mathbb{C}_p\}$ be a rigid meromorphic cocycle.

Fact: If $\tau \in \iota(K) \cap \mathcal{H}_p$ an RM pt, then
$\text{Stab}_\Gamma(\tau) = \langle \gamma_\tau \rangle$ is cyclic.

$\rightsquigarrow$ Can consider $J(\gamma_\tau) \in \mathcal{M}^\times$, and

$$J[\tau] := [J(\gamma_\tau)](\tau) \in \mathbb{C}_p.$$

Lemma: $J[\tau]$ depends only on the class of J in $H^1_f(\Gamma, \mathcal{M}^\times)$.

Proof: If J' is a different representative, then $J' = J \cdot c$, c a coboundary, i.e. $\exists f \in \mathcal{M}^\times$ s.t.

$$c(\gamma) = {}^{\gamma}f / f.$$

So

$$\begin{aligned} J'[\tau] &= (J \cdot c)(\gamma_\tau)(\tau) \\ &= J(\gamma_\tau)(\tau) \cdot {}^{\gamma_\tau}f/f(\tau) = J[\tau] \cdot \frac{f(\gamma_\tau \tau)}{f(\tau)} \\ &= J[\tau]. \end{aligned}$$

$\square$

Conjecture (Darmon–Vonk). Let J := rigid meromorphic period function, $\tau \in \mathcal{H}_p$ an RM pt. Then:

(1) $J[\tau] \in \overline{\mathbb{Q}}$ is algebraic,

(2) $K(J[\tau])/K$ is abelian, where $K = \mathbb{Q}(\tau)$ is the associated real quadratic field.

(Actually much more precise: the conjecture says $J[\tau] \in \underbrace{H_\tau H_J}_{\text{finite composition of ring class fields}}$

§5: Examples

Note:

$$\mathrm{div}(j) \subset SL_2(\mathbb{Z}) \cdot \frac{1+\sqrt{-3}}{2},$$

where

$\frac{1+\sqrt{-3}}{2}$ = CM pt of smallest discriminant.

Darmon–Vonk: $\varphi = \frac{1+\sqrt{5}}{2}$ = RM pt of smallest discriminant. $\exists$ natural!! rigid meromorphic cocycle J_φ,

s.t. $J_\varphi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has divisor supp. in $\Gamma \cdot \varphi$

$\rightsquigarrow$ "$J_\varphi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a convincing RM analogue of j."

(1) $\tau = 2\sqrt{2}$, $H_\tau = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$.

- $p = 3$: $J_\varphi[2\sqrt{2}] = \frac{33 + 56\sqrt{-1}}{5 \cdot 13}$
- $p = 13$: $J_\varphi[2\sqrt{2}] = \frac{1 + 2\sqrt{-2}}{3}$

$\Bigg\}$ 100 digits p-adic accuracy

(2) $\tau = 2\sqrt{6}$, $H_\tau = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-1})$.

- $p=7$: $J_\varphi[2\sqrt{6}] = \dfrac{3 + 8\sqrt{2} + 12\sqrt{-1} + 2\sqrt{-2}}{17}$
- $p=17$: $J_\varphi[2\sqrt{6}] = \dfrac{2 + \sqrt{-3} + 3\sqrt{2} + 2\sqrt{-6}}{7}$

} 400 digit accuracy

<u>Remark:</u> ... OBSERVE! If p, q both inert, then p shows up in denominator of q-adic $J[\tau]$ $\iff$ q does in p-adic $J[\tau]$!

<u>Theorem:</u> (Darmon–Vonk). Let $\tau \in \mathcal{H}_p \cap \iota(K)$ an RM point. $\exists$ canonical class

$$[J_\tau] \in H^1_f(\Gamma, \mathcal{M}^\times)$$

associated to τ. Moreover, "this accounts for all of $H^1_f(\Gamma, \mathcal{M}^\times)$". [Hecke translates of generalisations of ...]

(How?? Let $\mathcal{R}^\times = \left\{ J\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : J \in H^1_f(\Gamma, \mathcal{M}^\times) \right\} \subset \mathcal{M}^\times$

= rigid meromorphic period functions.

D-V classify $\mathcal{R}^\times$ in terms of functions

$$j_\tau(z) := \prod_{\{w \in \Gamma\tau \,:\, N_{K/\mathbb{Q}}(w) < 0\}} \left[\frac{t_w(z)}{t_{pw}(z)} \right]^{\mathrm{sgn}(w)},$$

where $t_w(z) = \begin{cases} z - w & : |w| \le 1 \\ z/w - 1 & : |w| > 1. \end{cases}$ → then show $\exists\, J_\tau$ s.t. $J_\tau\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = j_\tau$.)