Real Quadratic Singular Moduli

Recap of what we've seen so far:

CM theory	RM theory
$K = \mathbb{Q}(\tau)$ imag. quad. field.	$K = \mathbb{Q}(\tau)$ real quad field, p inert
$\mathcal{H} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$	$\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$
$\tau \in \mathcal{H}$ CM pt	$\tau \in \mathcal{H}_0$ RM point
$SL_2(\mathbb{Z}) \circlearrowright \mathcal{H}$	$\Gamma := SL_2(\mathbb{Z}[\frac{1}{p}]) \circlearrowright \mathcal{H}_p$ ... $SL_2(\mathbb{Z})$ is too small! ... but this is too big...
Heegner pt $(\mathbb{C}/\mathcal{O}_\tau, \mathfrak{n}^{-1}/\mathcal{O}_\tau) \in X_0(N) \to E(\mathbb{Q}(\tau))$	Stark-Heegner pt [fill in later]
$j = SL_2(\mathbb{Z})$-inv. mero. fn on $\mathcal{H}$	?
$j(\tau) \in$ abelian extn H_τ of K	
$p \mid j(\tau_1) - j(\tau_2) \Rightarrow p \mid \frac{D_1 D_2 - x^2}{4} > 0$	

§1: Rigid meromorphic cocycles

Want: analogue of $j := SL_2(\mathbb{Z})$-invariant mero. fn on $\mathcal{H}$.

Guess: find $J =$ an $SL_2(\mathbb{Z}[\frac{1}{p}])$-inv. rigid mero. fn on $\mathcal{H}_p$.

... but $SL_2(\mathbb{Z}[\frac{1}{p}])$-invariance is much too strong:

$\tau \in \mathcal{H}_p \rightsquigarrow SL_2(\mathbb{Z}[\frac{1}{p}]) \cdot \tau$ is dense in $\mathcal{H}_p$ (rigid topology)

$\rightsquigarrow J$ is constant on a dense set

$\longrightarrow J$ is constant!!

Reinterpretation: $\mathcal{M}_{\mathbb{C}}^\times$:= meromorphic fns on $\mathcal{H}$,

$\mathcal{M}^\times$:= rigid meromorphic fns on $\mathcal{H}_p$

Then $j \in H^0(SL_2(\mathbb{Z}), \mathcal{M}_{\mathbb{C}}^\times)$,

& $H^0(\Gamma, \mathcal{M}^\times) = \mathbb{C}_p^\times$ constants.

Darmon–Vonk: Consider $H^1(\Gamma, \mathcal{M}^\times)$ instead!

↳ Same approach as for Stark–Heegner pts

Def'n: Let $\Gamma_\infty \subset \Gamma$ be the subgroup of upper-triangular matrices. Let

$$H^1_f(\Gamma, \mathcal{M}^\times) := \ker\left(\text{res}: H^1(\Gamma, \mathcal{M}^\times) \to H^1(\Gamma_\infty, \mathcal{M}^\times)/\mathbb{C}_p^\times\right)$$

the subgroup of quasi-parabolic classes.

Def'n: A rigid meromorphic cocycle is a cocycle $J: \Gamma \to \mathcal{M}^\times$ s.t.

$$[J] \in H^1_f(\Gamma, \mathcal{M}^\times).$$

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Explicitly: J is a function

$$J: \Gamma \to \mathcal{M}^\times$$

s.t.:

1) $J(\gamma_1\gamma_2) = J(\gamma_1) \times \gamma_1 J(\gamma_2) \quad \forall \gamma_1, \gamma_2 \in \Gamma$,

2) $J|_{\Gamma_\infty}$ is constant, ie. $\exists\, \alpha \in \mathbb{C}_p^\times$ s.t. $J(\gamma) = \alpha \quad \forall\, \alpha \in \Gamma_\infty$.

Q: Given such a J, how do we get "canonical" numbers attached to an RM pt $\tau \in \mathcal{H}_p$?

Lemma: $\mathrm{Stab}_\Gamma(\tau)$ is cyclic rank 1, ie. $\exists\, \gamma_\tau \in \Gamma$ s.t. $\mathrm{Stab}_\Gamma(\tau) = \langle \gamma_\tau \rangle$.

$\rightsquigarrow$ can consider $J(\gamma_\tau) \in \mathcal{M}^\times$, and then

$$J[\tau] := J(\gamma_\tau)(\tau) \in \mathbb{C}_p \cup \{\infty\}.$$

<u>Lemma:</u> $J[\tau]$ depends only on the class of J in $H^1_f(\Gamma, \mathcal{M}^\times)$.

Proof: If J' is a different representative, then $J' = J \cdot c$, c a coboundary, i.e. $\exists f \in \mathcal{M}^\times$ s.t.

$$c(\gamma) = {}^{\gamma}f / f.$$

So

$$\begin{aligned} J'[\tau] &= (J\cdot c)(\gamma_\tau)(\tau) \\ &= J(\gamma_\tau)(\tau) \cdot {}^{\gamma_\tau}f/f\,(\tau) = J[\tau] \cdot \frac{f(\gamma_\tau \tau)}{f(\tau)} \\ &= J[\tau]. \end{aligned}$$

$\square$

<u>Conjecture:</u> (Darmon–Vonk) J rigid mero. cocycle, τ RM pt, $K = \mathbb{Q}(\tau)$. Then $J[\tau]$ is algebraic. Moreover

$$J[\tau] \in H_J H_\tau.$$

<u>Here:</u> - H_τ := ring class field attached to τ

= abelian extn of K w/ $\mathrm{Gal}(H_\tau/K) \simeq \mathrm{Pic}^+(\mathcal{O}_\tau)$,

$$\mathcal{O}_\tau := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}[\tfrac{1}{p}]) : a\tau + b = c\tau^2 + d \right\}$$

$$\hookrightarrow K \quad \text{via} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto c\tau + d$$

numerical evidence suggests H_J is not necessary...

- H_J = "field of def'n of J" = finite composition of H_σ's, $\sigma \in \mathrm{Poles}\left(J\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$.

... remarkable thing: <u>algebraic</u>, in a naturally occurring way.

§2: Rigid meromorphic period functions

Q: How do we classify Γ-mero. cocycles?

Def'n: Let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$. Let

$$\mathcal{R}^\times := \left\{ J(S) : [J] \in H^1_f(\Gamma, \mathcal{M}^\times),\ J|_{\Gamma_\infty} \in \mathbb{C}_p^\times \right\} \subset \mathcal{M}^\times.$$

If $f \in \mathcal{R}^\times$, we call it a rigid meromorphic period function.

Fact: elements in $\mathcal{R}^\times$ satisfy simple relations:

$$j(-\tfrac{1}{z}) = j(z)^{-1}, \qquad j(p^2 z) = j(z), \qquad \frac{j(z+1)}{j(z)} = j\left(-\frac{z+1}{z}\right).$$

Idea: classify $\mathcal{R}^\times$, and use this to describe $\mathcal{M}^\times$.

$\longrightarrow$ Closely parallel to a classical theory of rational period functions.

Example: 1) τ RM point,

$$\Sigma_\tau := \left\{ w \in \Gamma\tau : N_{K/\mathbb{Q}}(w) < 0 \right\} \subset \Gamma\tau \text{ discrete.}$$

(contains only finitely many $\mathbb{Z}$-translates).

2) If $w \in \Sigma_\tau$, let

$$t_w(z) := \begin{cases} z - w & : |w| \le 1 \\ z/w - 1 & : |w| > 1 \end{cases}.$$

3) Let

$$j_\tau^+(z) := \prod_{w \in \Sigma_\tau} \left(\frac{t_w(z)}{t_{pw}(z)} \right)^{\operatorname{sgn}(w)}.$$

Theorem: $j_\tau^+(z)$ converges to a rigid meromorphic period function. In particular, $\exists$

$$J_\tau^+ \in H_f^1(\Gamma, \mathcal{M}^\times)$$

with $J_\tau^+(S) = j_\tau^+ \in \mathcal{R}^\times \subset \mathcal{M}^\times$.

Darmon-Vonk: give complete classification of $\mathcal{R}^\times = \{$ linear combinations of Hecke translates of generalizations of $j_\tau^+ \}$.

+ give algorithm for computing J_τ^+.

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§3: Explicit examples

Let $\varphi := \frac{1+\sqrt{5}}{2}$ = RM pt of smallest discriminant.

$$\rightsquigarrow j_\varphi^+ : \mathcal{H}_p \longrightarrow \mathbb{C}_p \cup \{\infty\} \quad \in \mathcal{R}^\times \subset \mathcal{M}^\times$$

$$\rightsquigarrow J_\varphi^+ \in H_f^1(\Gamma, \mathcal{M}^\times), \qquad j_\varphi^+ = J_\varphi^+\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

Remark: - divisor of j_φ^+ concentrated in $\Gamma \varphi$.

- divisor of j is concentrated in $SL_2(\mathbb{Z}) \cdot \frac{1+\sqrt{-3}}{2}$

- φ = RM pt of smallest discriminant, $\frac{1+\sqrt{-3}}{2}$ = CM pt of smallest disc.

$\rightsquigarrow$ "$j_\varphi^+ : \mathcal{H}_p \longrightarrow \mathbb{C}_p \cup \{\infty\}$ is a convincing analogue of $j : \mathcal{H} \to \mathbb{C} \cup \{\infty\}$".

Examples:

(1) Let $K = \mathbb{Q}(\sqrt{2})$, $\tau = 2\sqrt{2} \rightsquigarrow H_\tau = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$.

- $p = 3$: $J_\varphi^+[2\sqrt{2}] \stackrel{?}{=} \frac{33 + 56\sqrt{-1}}{5 \cdot 13}$

- $p = 13$: $J_\varphi^+[2\sqrt{2}] \stackrel{?}{=} \frac{1 + 2\sqrt{-2}}{3}$

} 100 digits of p-adic accuracy.

(5 not inert in $\mathbb{Q}(\sqrt{2})$).

(2) $K = \mathbb{Q}(\sqrt{6})$, $\tau = 2\sqrt{6}$, $H_\tau = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-1})$.

- $p = 7$: $$J_\varphi^+[2\sqrt{6}] = \frac{3 + 8\sqrt{2} + 12\sqrt{-1} + 2\sqrt{-2}}{17} \pmod{7^{400}}$$

- $p = 17$: $$J_\varphi^+[2\sqrt{6}] = \frac{2 + \sqrt{-3} + 3\sqrt{2} + 2\sqrt{-6}}{7} \pmod{17^{400}}.$$

(3) [Share table: p42 of Darmon-Vonk].

$\longrightarrow$ coeff of φ is appearing in H_{ne};

$\rightsquigarrow$ primes dividing denominators $J[\tau]$ are small;

$\rightsquigarrow$ if $p \mid$ (denom. of q-adic $J_\varphi[\tau]$), then often

$q \mid$ (denom. of p-adic $J_\varphi[\tau]$) !!!

§4: p-adic intersection numbers

Finally: analogue of Gross-Zagier's thm on $J_\infty(\tau_1, \tau_2) := j(\tau_1) - j(\tau_2)$.

Definition: $\tau_1, \tau_2 \in \mathcal{H}_p$ RM pts. Let

$$J_p(\tau_1, \tau_2) := \widehat{J}_{\tau_1}[\tau_2].$$

Thm 3.20 of D-V. modification of $J_{\tau_1}^+$: $J_{\tau_1}^+ \longrightarrow \overline{J_\tau}/\overline{J_{p\tau}}$

under $H^1_f(\Gamma, \mathcal{M}^\times) \longrightarrow H^1_{par}(\Gamma, \mathcal{M}^\times/K_p^\times)$

<u>Conjecture</u>: (i) $J_p(\tau_1,\tau_2) \in H_{\tau_1} H_{\tau_2}$.

(ii) If $J_p(\tau_1,\tau_2) \equiv 0 \pmod{q}$, then q does not split in $\mathcal{O}(\tau_1)$ or $\mathcal{O}(\tau_2)$, and

$$q \,\Big|\, \underbrace{\frac{D_1 D_2 - x^2}{4}}_{} > 0$$

for some $x \in \mathbb{Z}$.

(iii) $V_q\big(J_p(\tau_1,\tau_2)\big) \sim$ "weighted intersection number" on a Shimura curve for $B :=$ quaternion algebra ramified only at p, q.

switching p and q doesn't change this!

so $V_p J_q(\tau_1,\tau_2) = V_q J_p(\tau_1,\tau_2)$.