On p-adic L-functions for GL_{2n} in finite slope Shalika families

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Authors' note

This paper has been accepted for publication in Adv. Math. *The first version of the paper had additional details that we stripped out when revising for publication. This file has exactly the same results as the accepted version, with the same theorem/lemma/equation numbering; but it contains more details and motivation throughout.*

Abstract

In this paper, we propose and explore a new connection in the study of p-adic L-functions and eigenvarieties. We use it to prove results on the geometry of the cuspidal eigenvariety for GL_{2n} over a totally real number field F at classical points admitting Shalika models. We also construct p-adic L-functions over the eigenvariety around these points. Our proofs proceed in the opposite direction to established methods: rather than using the geometry of eigenvarieties to deduce results about p-adic L-functions, we instead show that non-vanishing of a (standard) p-adic L-function implies smoothness of the eigenvariety at such points. Key to our methods are a family of distribution-valued functionals on (parahoric) overconvergent cohomology groups, which we construct via p-adic interpolation of classical representation-theoretic branching laws for $\mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n}$.

More precisely, we use our functionals to attach a p-adic L-function to a non-critical refinement $\tilde{\pi}$ of a regular algebraic cuspidal automorphic representation π of GL_{2n}/F which is spherical at p and admits a Shalika model. Our new parahoric distribution coefficients allow us to obtain optimal non-critical slope and growth bounds for this construction. When π has regular weight and the corresponding p-adic Galois representation is irreducible, we exploit non-vanishing of our functionals to show that the parabolic eigenvariety for GL_{2n}/F is étale at $\tilde{\pi}$ over an $([F:\mathbb{Q}]+1)$-dimensional weight space and contains a dense set of classical points admitting Shalika models. Under a hypothesis on the local Shalika models at bad places which is empty for π of level 1, we construct a p-adic L-function for the family.

Contents

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1. Introduction

The arithmetic of L-functions has long been a topic of intense interest in number theory. Via the Bloch–Kato Conjecture, the special values of L-functions are expected to carry deep algebraic data. Most recent progress towards this conjecture has come through p-adic methods – more precisely, through understanding all of (a) p-adic L-functions, (b) classical p-adic families, and (c) p-adic L-functions over classical p-adic families. Where one has all three, they have been crucial in proofs of Iwasawa Main Conjectures and cases of the Bloch–Kato Conjecture. It is therefore natural to ask whether one can obtain (a), (b) and (c) for any regular algebraic cuspidal automorphic representation (RACAR) of a reductive group G.

For (a), at least, this is expected to be possible in great generality, thanks to conjectures of Coates–Perrin-Riou and Panchishkin [36, 35, 66]. However, our understanding of fundamental cases – for example, GL_N for $N > 2$ – remains poor, with relatively few constructions of p-adic L-functions in this case, most of which assume a p-ordinarity condition.

The theory of p-adic families is more subtle still. Singularly, (b) can *fail* when moving beyond GL_2; there exist RACARs of GL_N that are 'arithmetically rigid', not varying in any classical p-adic family (that is, a positive-dimensional subspace of an eigenvariety containing a Zariski-dense set of classical points; see e.g. [8]). This contrasts sharply with the cases of, for example, Hilbert or Siegel modular forms, where it is expected all RACARs can be classically varied.

To approach (c), one needs not only the existence of a classical family, but also a precise description of its geometry. For example, one needs to know whether such a family is smooth or étale over the weight space. Well-established methods for studying eigenvarieties break down for GL_N with $N > 2$, owing to RACARs contributing to multiple degrees of cohomology and the underlying locally symmetric spaces not admitting any algebraic structure. The geometry of the GL_N eigenvariety is thus largely mysterious, meaning there are few instances in this setting where (c) is approachable at present.

In this paper, we prove new cases of (a), (b) and (c) for regular algebraic *symplectic-type* cuspidal automorphic representations (RASCARs) π of GL_N over a totally real number field F, described below in Theorems A, B and C respectively.

The technical heart of our approach is a construction of 'evaluation maps', a p-adic integration theory on overconvergent modular symbols for GL_N. The special values of these maps compute explicit multiples of classical complex L-values of RASCARs. Such maps are very familiar in the setting of GL_2, where they have been used in many papers to study p-adic L-functions (see §1.2.1), but they had not previously been constructed for any higher-dimensional GL_N. The GL_2 constructions do not easily generalise; the relative simplicity of the GL_2 setting hides substantial representation-theoretic obstructions that arise in higher dimension (see §1.2.2). A key new input in our constructions is a p-adic interpolation, in both cyclotomic and weight directions, of higher-dimensional branching laws in representation theory. This occupies §5 and §6.

Once constructed, evaluation maps have powerful consequences. Their utility in constructing p-adic L-functions is already well-documented in the GL_2 case, and similarly we use them to construct p-adic L-functions for RASCARs of GL_N. However, we also push their use further than previous works. One particularly striking consequence is the following strong version of (b) in this setting, made precise in Theorem B (and Theorem 7.6):

(†) *Let π be a RASCAR with regular weight and irreducible Galois representation. Then the parabolic GL_N-eigenvariety is étale over the pure weight space at certain non-critical p-refinements $\tilde{\pi}$ of π. Hence $\tilde{\pi}$ varies in a unique classical p-adic family.*

Our proof of this result turns traditional methods upside down. There is a long and storied history of applying the geometry of eigenvarieties to construct and study p-adic L-functions; for example, this is the central tenet of Bellaïche's celebrated paper on critical p-adic L-functions [20]. There are also some works connecting the étaleness of an eigenvariety to the non-vanishing of an *adjoint* p-adic L-function (e.g. [21, §VIII], [10], [57]), but similarly all of these works require prior knowledge about existence and properties of p-adic families to study the p-adic L-function.

Our methods add to this rich story. However, they differ considerably in that unlike all previous works, we proceed in the opposite direction. *We first construct p-adic L-functions and then we use them to construct p-adic families.* Indeed, we use our evaluation maps to show that non-vanishing of the p-adic L-function of $\tilde{\pi}$ – guaranteed by regular weight – implies faithfulness of a Hecke algebra as a module over weight space, thus producing dimension in the eigenvariety and implying the existence of classical families. In addition, rather than using the adjoint p-adic

L-function, to our knowledge we give the first instance where non-vanishing of a *standard* p-adic L-function is used to control the geometry of an eigenvariety.

The methods we develop in this paper have more general applications. In particular, the proof of (†) – which occupies all of §7 – shows that evaluation maps, through interpolation of branching laws, can be a powerful tool in understanding the geometry of classical p-adic families. We have explored this further in sequel papers [13, 14]. More generally, our methods suggest the natural setting to consider evaluation maps is that of spherical varieties, giving strong connections between the geometry of eigenvarieties and automorphic period integrals in the Gan–Gross–Prasad conjectures (see e.g. [87, 67, 88]), as well as to p-adic interpolation in Sakellaridis–Venkatesh's relative Langlands program [72]. We will use the methods of this paper to construct and study p-adic interpolations of such period integrals in future work.

We expect there to be further arithmetic applications of our evaluation maps. In the GL_2 setting, beyond their applications to p-adic L-functions, analogues of these maps have further been used to study periods and congruences between base-change and non-base-change Bianchi modular forms [49, 79], study $\mathcal{L}$-invariants and trivial zero conjectures [15, 11], construct Stark–Heegner cycles predicted by the Bloch–Kato conjecture [84, 83], and prove generalisations of Hida duality [24]. Few of these results/constructions have been carried out for higher-dimensional GL_N, by any method. We anticipate similar applications of evaluation maps are possible for RASCARs, and again hope to return to this in subsequent work.

Finally, we mention applications of our results themselves, which – as explained above – should ultimately have applications towards the Bloch–Kato and Iwasawa Main Conjectures. They have already led to research in this direction for GL_{2n} [70, 58]. There are more immediate applications to other groups such as GSp_4. In a sequel [13] to this paper, we have crucially used the methods developed here to prove a result on the variation of p-adic L-functions required in [60]. This result – which was announced as Theorem 17.6.2 *ibid.*, where it was deferred to future work of the present authors – was used by Loeffler and Zerbes to prove cases of the Bloch–Kato Conjecture for GSp_4 [60] and for symmetric cubes of modular forms [61].

1.1. Set-up and previous work. Fix forever an isomorphism $i_p : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$.

We say a regular algebraic cuspidal automorphic representation (RACAR) π of $\mathrm{GL}_N(\mathbf{A}_F)$ is *essentially self-dual* if $\pi^\vee \cong \pi \otimes \eta^{-1}$ for a Hecke character η, which we henceforth fix. Such a π is either η-*orthogonal* or η-*symplectic*, depending respectively on whether the twisted symmetric square L-function $L(\mathrm{Sym}^2\pi \times \eta^{-1}, s)$ or the twisted exterior square L-function $L(\bigwedge^2 \pi \times \eta^{-1}, s)$ has a pole at $s = 1$ (see [62, 75], summarised in detail in [50, Lem. 2.1]). Assuming the p-adic Galois representation ρ_π attached to an essentially self-dual RACAR π is (absolutely) irreducible, the unique (up to scalar) non-degenerate η-equivariant bilinear form on ρ_π is either symmetric or skew-symmetric, and it is conjectured that π should be η-orthogonal in the former case and η-symplectic in latter. Our focus is on the η-symplectic case. By [4], π is η-symplectic if and only if $N = 2n$ is even and either (so both) of the following hold:

(i) π admits an (η, ψ)-Shalika model (see §2.6);

(ii) π is the transfer of a globally generic cuspidal automorphic representation Π of $\mathrm{GSpin}_{2n+1}(\mathbf{A}_F)$ with central character η.

Henceforth we will mainly use (i) and call such a representation π a *RASCAR*.

Let $G = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\,\mathrm{GL}_{2n}$. Let Σ be the set of real embeddings of F, and let $\lambda = (\lambda_\sigma)_{\sigma\in\Sigma}$ be a Borel-dominant weight for G, i.e. $\lambda_\sigma = (\lambda_{\sigma,1}, \dots, \lambda_{\sigma,2n}) \in \mathbf{Z}^{2n}$ with $\lambda_{\sigma,1} \geqslant \cdots \geqslant \lambda_{\sigma,2n}$. Let π be a RASCAR of $G(\mathbf{A})$ of weight λ; our convention is that π is cohomological with respect to the coefficient system $V_\lambda^\vee$, where V_λ is the algebraic representation of highest weight λ. Let

$$\mathrm{Crit}(\lambda) := \{j \in \mathbf{Z} : -\lambda_{\sigma,n} \leqslant j \leqslant -\lambda_{\sigma,n+1}\ \forall \sigma \in \Sigma\}. \tag{1.1}$$

Then $j \in \mathrm{Crit}(\lambda)$ if and only if $L(\pi, j+\frac{1}{2})$ is a Deligne-critical value; and Grobner–Raghuram showed in [45] that these L-values, and their twists by finite order Hecke characters, are algebraic multiples of a finite set of complex periods.

Let p be a prime such that $\pi_{\mathfrak{p}}$ is spherical for each $\mathfrak{p}|p$ (or more generally, each $\pi_{\mathfrak{p}}$ satisfies (C2) of Conditions 2.8). A p-adic L-function for π is a p-adic distribution of controlled growth that interpolates the algebraic parts of Deligne-critical L-values. For such p-adic interpolation it is essential to take a p-refinement $\tilde{\pi}$ of π, i.e. to work at non-maximal level at p (e.g. for GL_2, this is the process of passing from a newform of level $\Gamma_1(M)$ to an eigenform of level $\Gamma_1(M) \cap \Gamma_0(p)$). A standard approach is to refine to Iwahori level at $\mathfrak{p}|p$, which in our case corresponds to choosing a full triangulation of the $2n$-dimensional local Galois representation. However, the Panchishkin condition (see [59, §2.1], inspired by [66]) predicts that the p-adic L-function should not depend on a full triangulation, but only on a suitable n-dimensional stable submodule. This suggests that the natural level to take at $\mathfrak{p}|p$ is not Iwahoric, but the parahoric subgroup $J_{\mathfrak{p}}$ relative to the parabolic subgroup Q of G with Levi

$$H = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}(\mathrm{GL}_n \times \mathrm{GL}_n).$$

In this paper, we indeed show that parahoric level is optimal for this construction (see §1.2.3).

Let $\mathcal{O}_{\mathfrak{p}}$ be the ring of integers in the completion $F_{\mathfrak{p}}$ of F at $\mathfrak{p}$, and fix a uniformiser $\varpi_{\mathfrak{p}}$ in $\mathcal{O}_{\mathfrak{p}}$. Of central importance is the Hecke operator $U_{\mathfrak{p}} = \left[J_{\mathfrak{p}} \begin{pmatrix} \varpi_{\mathfrak{p}} I_n & \\ & I_n \end{pmatrix} J_{\mathfrak{p}}\right]$ and its optimal integral normalisation $U_{\mathfrak{p}}^{\circ}$ (see §3.3). A *Q-refinement of* $\pi_{\mathfrak{p}}$ is a choice of (non-zero) eigenvalue $\alpha_{\mathfrak{p}}$ of $U_{\mathfrak{p}}$ acting on $\pi_{\mathfrak{p}}^{J_{\mathfrak{p}}}$, and a *$Q$-refinement of* π is a choice $\tilde{\pi} = (\pi, (\alpha_{\mathfrak{p}})_{\mathfrak{p}|p})$ of Q-refinement of $\pi_{\mathfrak{p}}$ for each $\mathfrak{p}|p$. Following [38, Definition 3.5], we say the Q-refinement $\tilde{\pi}$ is *Shalika* if each $\alpha_{\mathfrak{p}}$ is a simple eigenvalue that interacts well with the Shalika model in a precise sense (see §2.7).

The following p-adic reformulation of [45] performed in [38] will be crucial for us. For an open compact subgroup $K = \prod_v K_v \subset G(\mathbf{A}_f)$ that is *parahoric at* p (that is, with $K_{\mathfrak{p}} = J_{\mathfrak{p}}$ at each $\mathfrak{p}|p$), let S_K be the associated locally symmetric space for G (see (2.3)). Consider the compactly supported cohomology groups $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))$ in degree $t = d(n^2+n-1)$, where $\mathscr{V}_\lambda^\vee$ is the (p-adic) algebraic local system attached to $V_\lambda^\vee$. Then [38] proved that

(a) there exists a family of p-adic classical evaluation maps

$$\mathrm{Ev}_{\chi,j} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p)) \to \overline{\mathbf{Q}}_p,$$

indexed by finite order Hecke characters χ of p-power conductor and $j \in \mathrm{Crit}(\lambda)$, and

(b) for a sufficiently small but inexplicit level $K = K(\tilde{\pi})$ (see (2.23)), attached to $\tilde{\pi}$ and i_p there exists a classical eigenclass

$$\phi_{\tilde{\pi}} \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))$$

whose image $\mathrm{Ev}_{\chi,j}(\phi_{\tilde{\pi}})$ equals $i_p\left(L(\pi \otimes \chi, j+\frac{1}{2})/\Omega_\pi^\epsilon\right)$ up to a non-zero factor, where Ω_π^ϵ is a complex period, depending only on π and the multi-sign $\epsilon = (-1)^j \cdot (\chi\eta)_\infty \in \{\pm 1\}^\Sigma$ (recalling π is η-symplectic).

In Theorem 5.22, based on [54], we improve this by making the factor completely explicit.

In [38], when further $\tilde{\pi}$ is *Q-ordinary* – that is, when the integral normalisation $\alpha_{\mathfrak{p}}^{\circ}$ of $\alpha_{\mathfrak{p}}$ is a p-adic unit – the authors constructed the corresponding p-adic L-function $\mathcal{L}_p(\tilde{\pi})$ by proving directly the so-called Manin relations.

1.2. Main results and methods. In this section, we state our three main results precisely (in Theorems A, B and C). All three are proved using *overconvergent cohomology.*

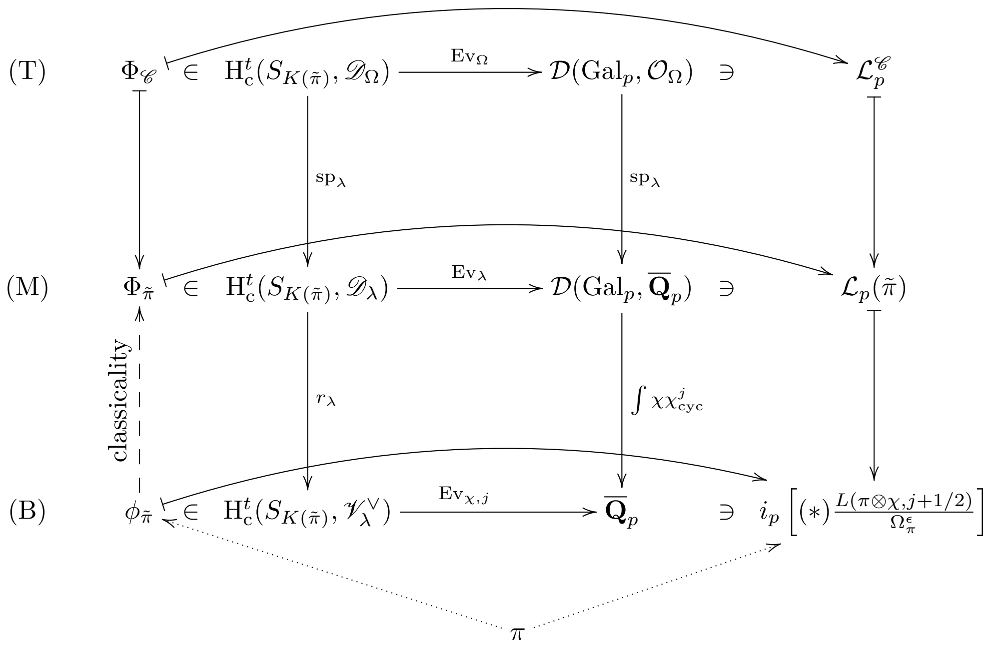

Figure 1: *Strategy of our constructions*

1.2.1. Overconvergent cohomology and families beyond GL_2*.* The utility of overconvergent cohomology in constructing p-adic L-functions and p-adic families is very familiar. For example,

(1) it was the central method used in the papers [43, 68, 69, 20, 12, 86, 16, 11, 23, 17, 18, 56], including several by the present authors, to construct and study p-adic L-functions attached to modular forms on GL_2 (in various settings, ranging from ordinary modular forms over $\mathbf{Q}$ to finite slope families over general number fields);

(2) for a general quasi-split group G, it was used in [9, 81, 46, 19] to construct eigenvarieties (and hence p-adic families of overconvergent systems of eigenvalues).

Despite the huge generality for which overconvergent cohomology has been developed in (2), none of the program in (1) has been generalised from GL_2 to higher GL_N. From the papers above the *strategy* for carrying out such a generalisation is clear; it is summarised in Figure 1 below. The results of [45, 38] comprise the bottom row (B). However, fundamental obstacles and new features arise when trying to implement this strategy to construct the middle and top row beyond GL_2. For example, we have already commented that the theory of p-adic families (and hence row (T)) is not well-understood. Additionally, unlike for GL_2 there is subtlety over the level at p (Iwahoric vs. parahoric) at which one works; we describe this in detail in §1.2.3 below.

Before any of this, however, one must first construct (horizontal) *evaluation maps* in Figure 1. In row (B), where there is no p-adic variation, the evaluation maps $\mathrm{Ev}_{\chi,j}$ depend on a separate choice of classical representation-theoretic branching law for each $j \in \mathrm{Crit}(\lambda)$. For GL_2, the classical coefficients $V_\lambda^\vee$ are just spaces of polynomial functions on $\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p$. This simplicity yields obvious canonical choices of branching laws, which are readily p-adically interpolated, making the construction of rows (M) and (T) straightforward.

For higher GL_N, the coefficient modules are hard to describe explicitly. Choices of branching laws are no longer canonical, and must be carefully aligned for p-adic interpolation to even be possible. A key technical result of [38] was Theorem 2.4.1, where Januszewski, Raghuram and one of us carried out such an alignment, using finite-dimensional coefficient modules, for fixed λ and j varying in a finite set. However, their method does not generalise to our infinite-dimensional distribution coefficients. In this paper, we develop a new way of aligning branching laws, sketched in §1.2.2 below, and use it to construct Ev_λ in our setting. We also explain how to further align these branching laws as λ varies in a family Ω, and use this to construct Ev_Ω.

1.2.2. p-adic L-functions for finite slope RASCARs. Our first main result is the construction of a p-adic L-function attached to a finite slope RASCAR of GL_{2n}. More precisely, as predicted by Panchishkin [66] we construct a p-adic distribution on Gal_p, the Galois group of the maximal abelian extension of F unramified outside $p\infty$, satisfying growth and interpolation properties.

Overconvergent cohomology groups are defined by replacing the algebraic coefficients $V_\lambda^\vee$ with spaces of p-adic distributions $\mathcal{D}_\lambda$ and $\mathcal{D}_\Omega$, where Ω is a (rigid analytic) family in which the weight λ varies. To work at Q-parahoric level, as mentioned above (see also §1.2.3 below), we use a new class of parahoric distributions from our companion paper [19], described here in §3. These spaces are constructed as a 'double induction': first we take an algebraic induction of λ to H, and then a locally analytic induction to the parahoric $J_p \subset G(\mathbf{Q}_p)$. These distributions are analytic along the unipotent radical of Q, but algebraic along all other variables (unlike Iwahoric distributions, which are analytic in all variables).

At any fixed λ, the space $\mathcal{D}_\lambda$ admits $V_\lambda^\vee$ as a quotient, inducing a specialisation map

$$r_\lambda : \mathrm{H}^\bullet_{\mathrm{c}}(S_{K(\tilde\pi)}, \mathscr{D}_\lambda) \to \mathrm{H}^\bullet_{\mathrm{c}}(S_{K(\tilde\pi)}, \mathscr{V}_\lambda^\vee).$$

We say the refinement $\tilde\pi = (\pi, \{\alpha_\mathfrak{p}\}_{\mathfrak{p}|p})$ is *non-Q-critical* if r_λ becomes an isomorphism after restricting to the generalised Hecke eigenspaces at $\tilde\pi$ (see Definition 3.14); this guaranteed by having *non-Q-critical slope* at p (Theorem 3.16). For non-Q-critical $\tilde\pi$, lifting $\phi_{\tilde\pi}$ under the isomorphism r_λ, we obtain a class $\Phi_{\tilde\pi} \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)}, \mathscr{D}_\lambda)$.

We now come to our major new input: the construction of a family of *evaluation maps* on overconvergent cohomology groups, comprising the horizontal maps in Figure 1 and occupying all of §4–§6. More precisely, we construct a map

$$\mathrm{Ev}_\lambda : \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)}, \mathcal{D}_\lambda)[U_\mathfrak{p}^\circ - \alpha_\mathfrak{p}^\circ : \mathfrak{p}|p] \longrightarrow \mathcal{D}(\mathrm{Gal}_p, \overline{\mathbf{Q}}_p)$$

on the $\{U_\mathfrak{p}^\circ = \alpha_\mathfrak{p}^\circ\}_{\mathfrak{p}|p}$ eigenspace, valued in a space $\mathcal{D}(\mathrm{Gal}_p, \overline{\mathbf{Q}}_p)$ of locally analytic distributions, which interpolates all of the $\mathrm{Ev}_{\chi,j}$ simultaneously as χ and j vary. We note that $\mathcal{D}(\mathrm{Gal}_p, \overline{\mathbf{Q}}_p)$ is the space which Panchishkin predicts should contain the p-adic L-function of $\tilde\pi$.

The existence of the map Ev_λ is a 'p-adic interpolation of branching laws' for $H \subset G$ that gives, for free, all of the classical Manin relations (as computed in [38]). More precisely, for $j_1, j_2 \in \mathbf{Z}$, let $V^H_{(j_1,j_2)}$ denote the H-representation $\det_1^{j_1}\det_2^{j_2}$, the algebraic representation of highest weight $(j_1, ..., j_1, j_2, ..., j_2)_{\sigma\in\Sigma}$. Let $\mathsf{w} \in \mathbf{Z}$ be the purity weight of λ (see §2.2). Then we have the following reinterpretation of the Deligne-critical L-values (1.1):

Branching law: $j \in \mathrm{Crit}(\lambda) \iff V^H_{(-j,\mathsf{w}+j)} \subset V_\lambda\big|_H$ *with multiplicity one.*

For each fixed $j \in \mathrm{Crit}(\lambda)$, the map $\mathrm{Ev}_{\chi,j}$ depends on a (non-canonical) choice of basis vector $v_j \in V^H_{(-j,\mathsf{w}+j)} \subset V_\lambda|_H$. To construct Ev_λ these must be carefully aligned. We do this in §5 by reinterpreting the module V_λ as a double algebraic induction, collapsing all choices onto a single choice of branching law for $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$ diagonally embedded inside H. In the process, we obtain an explicit description of the branching law for $H \subset G$.

In §6, we use our parahoric distributions to construct the required p-adic interpolation of the above branching laws in the cyclotomic direction. The parahoric setting allows us to isolate (in the first induction) the algebraic branching law for $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$ fixed above, whilst allowing the second induction to vary p-adic analytically. Essential for this interpolation is a family of support conditions for evaluation maps, arising from a choice of open orbit representative ξ for the spherical variety G/H. This is a representation-theoretic avatar of the familiar phenomenon in constructing p-adic L-functions, whereby one must modify the Euler factor at p.

Definition 1.1. Let π be a RASCAR of $G(\mathbf{A})$ spherical at p. Suppose π admits a non-Q-critical, Shalika Q-refinement $\tilde{\pi} = (\pi, (\alpha_{\mathfrak{p}})_{\mathfrak{p}|p})$, and let $\Phi_{\tilde{\pi}}$ be the resulting overconvergent lift of $\phi_{\tilde{\pi}}$, which is a $U_{\mathfrak{p}}^{\circ}$-eigenclass for all $\mathfrak{p}|p$. Define the *p-adic L-function* of $\tilde{\pi}$ to be

$$\mathcal{L}_p(\tilde{\pi}) := \mathrm{Ev}_\lambda(\Phi_{\tilde{\pi}}) \in \mathcal{D}(\mathrm{Gal}_p, \overline{\mathbf{Q}}_p).$$

This depends on $\phi_{\tilde{\pi}}$, hence (by (2.22)) on the restriction of i_p to the number field E; but this is an expected indeterminacy (corresponding to [35, (14)]) which we largely suppress.

Our first main result, proved in Theorem 6.23 and illustrated in the middle row (M) of Figure 1, is that the distribution $\mathcal{L}_p(\tilde{\pi})$ satisfies suitable growth and interpolation properties, justifying the terminology 'p-adic L-function'. Observe that finite order Hecke characters of p-power conductor, and the p-adic cyclotomic character χ_{cyc}, are characters of Gal_p (see §6.1.1).

Theorem A. *Let π and $\tilde{\pi}$ be as in Definition 1.1. Then:*

(1) $\mathcal{L}_p(\tilde{\pi})$ is admissible of growth $h_p := v_p\left(\prod_{\mathfrak{p}|p}(\alpha_{\mathfrak{p}}^{\circ})^{e_{\mathfrak{p}}}\right)$ (see Definition 6.19);

(2) for all finite order Hecke characters χ of conductor $\prod_{\mathfrak{p}|p}\mathfrak{p}^{\beta_{\mathfrak{p}}}$ and all $j \in \mathrm{Crit}(\lambda)$, we have

$$i_p^{-1}(\mathcal{L}_p(\tilde{\pi}, \chi\chi_{\mathrm{cyc}}^j)) = A \cdot \tau(\chi_f)^n \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d})^{jn} \prod_{\mathfrak{p}|p} e_{\mathfrak{p}}(\tilde{\pi}, \chi, j) \cdot e_\infty(\pi, \chi, j) \cdot \frac{L^{(p)}\left(\pi \otimes \chi, j + \frac{1}{2}\right)}{\Omega_\pi^\epsilon},$$

where $e_{\mathfrak{p}}(\tilde{\pi}, \chi, j)$ is the Coates–Perrin-Riou factor at $\mathfrak{p}$ (defined in Theorem 6.23), $e_\infty(\pi, \chi, j)$ is the modified Euler factor at infinity (Definition 5.18), $A \in \mathbf{Q}^\times$ is a constant (6.25), *$\mathfrak{d}$ is the different of $F/\mathbf{Q}$, $\epsilon = (\chi\chi_{\mathrm{cyc}}^j\eta)_\infty \in \{\pm 1\}^\Sigma$, $\tau(\chi_f)$ is the Gauss sum, $L^{(p)}(-)$ is the (finite) L-function without factors at p, and Ω_π^ϵ is a complex period.*

If $h_p < \#\mathrm{Crit}(\lambda)$, then the restriction of $\mathcal{L}_p(\tilde{\pi})$ to the cyclotomic line is unique with these properties; if further Leopoldt's conjecture holds for F at p, then $\mathcal{L}_p(\tilde{\pi})$ itself is unique.

Remark 1.2. Note that, exploiting work of Jiang–Sun–Tian [54], we are able to prove the full expected period relations at infinity. As a consequence, both sides of (2) lie in an explicit number field $E(\chi)$, with E defined in §2.9. Moreover $\mathcal{L}_p(\tilde{\pi})$ can be taken with coefficients in a finite extension $L/\mathbf{Q}_p$ containing $i_p(E)$. We have suppressed this here to ease notation.

In the first draft of this paper, in the interpolation we restricted to characters ramified at all $\mathfrak{p}|p$; but in a separate paper [13] with Graham and Jorza, we computed the relevant unramified zeta integrals which – when combined with the construction here – give the full Coates–Perrin-Riou/Panchishkin conjecture in this case. Even in the ordinary case Theorem A upgrades [38]; indeed, it also corrects a small error in the interpolation formula *ibid.* (see Appendix (2)).

1.2.3. Benefits of the parahoric approach. Let us now precisely highlight the benefit of using parahoric (rather than Iwahori) distributions. Our primary motivation is the conjecture of Panchishkin [66, Conj. 6.2]; using our approach we prove exactly the automorphic version of his conjecture, including the growth/unicity bounds. These would not follow from Iwahori methods.

We illustrate via examples. Let π be a RASCAR of $\mathrm{GL}_4(\mathbf{A})$ spherical and regular at p. Then:

- The Iwahori-invariants π_p^{Iw} are 24-dimensional, a direct sum of 24 1-dimensional simultaneous eigenspaces for the Hecke operators $U_{p,1}, U_{p,2}, U_{p,3}$, with $U_{p,i}$ attached to $\begin{pmatrix} pI_i & \\ & I_{4-i}\end{pmatrix}$. An Iwahori refinement $\tilde{\pi}' = (\pi, \alpha_{p,1}, \alpha_{p,2}, \alpha_{p,3})$ is a choice of one of these 24 eigensystems.
- The Q-parahoric invariants $\pi_p^{J_p}$ are 6-dimensional, giving at most 6 Q-refinements $\tilde{\pi} = (\pi_p, \alpha_p)$, where α_p is a choice of U_p-eigenvalue on $\pi_p^{J_p}$.

When there are 6 distinct Q-refinements, above each such (π, α_p) there are 4 Iwahori refinements $\tilde{\pi}'_1, \tilde{\pi}'_2, \tilde{\pi}'_3, \tilde{\pi}'_4$, each with $\alpha_{p,2} = \alpha_p$.

To work at Iwahori level, we must choose an Iwahori refinement. To lift eigenclasses to overconvergent cohomology, we must control all of the (normalised) $U^\circ_{p,i}$ operators; for example, the non-critical slope bound depends on all three of the slopes $h_i := v_p(\alpha^\circ_{p,i}) \geqslant 0$. Working solely at Iwahori level, you bound the growth of the p-adic L-function only by the sum $h_1 + h_2 + h_3$.

By contrast, working at Q-parahoric level, lifting requires control only of U°_p from §1.1, the non-critical slope bound depends only on h_2, and we get growth bounded by h_2.

In this paper, and its sequel [13], we show that p-adic L-functions depend only on the parahoric refinement. In particular, in [13, §12.4, §14] we attach p-adic L-functions $\mathcal{L}_p(\tilde{\pi}'_i)$ to the four Iwahori refinements $\tilde{\pi}'_i$ above $\tilde{\pi}$ (under stronger hypotheses, and with ostensibly weaker growth) and compare to this paper to prove that $\mathcal{L}_p(\tilde{\pi}) = \mathcal{L}_p(\tilde{\pi}'_1) = \mathcal{L}_p(\tilde{\pi}'_2) = \mathcal{L}_p(\tilde{\pi}'_3) = \mathcal{L}_p(\tilde{\pi}'_4)$ (up to rational scalar). Thus the parahoric Q-refinement is the exact amount of data required to construct a p-adic L-function; passing to deeper level requires additional but redundant hypotheses.

We give two explicit examples from the tables at `smf.compositio.nl` (cf. [14, §7]).

- There is a unique RASCAR π of $\mathrm{GL}_4(\mathbf{A})$ of weight $\lambda = (12, 1, -1, -12)$ and level 1 (taking $j = 2, k = 14$ in the table). At $p = 11$, π_{11} is Q-ordinary, but not Iwahori-ordinary. Let $\tilde{\pi}$ be the unique Q-ordinary refinement. Using parahoric methods, we get an 11-adic L-function $\mathcal{L}_{11}(\tilde{\pi})$ and can prove it is a bounded measure on $\mathbf{Z}_{11}^\times$ (i.e. growth bounded by 0), that is uniquely determined by growth and interpolation.

  There are four Iwahori refinements $\tilde{\pi}'_i$ above $\tilde{\pi}$, all non-critical and regular. Using only Iwahori methods, we obtain four 11-adic L-functions $\mathcal{L}_{11}(\tilde{\pi}'_i)$, and can prove these are distributions on $\mathbf{Z}_{11}^\times$ with growth bounded by 22, 12, 12 and 2 respectively. Without further input all four of these might be unbounded, and three are not uniquely determined by interpolation and these growth bounds.

  Via this paper and [13], however, we know all of these 11-adic L-functions are in fact equal.
- There is a unique RASCAR of $\mathrm{GL}_4(\mathbf{A})$ of weight $(9, 6, -6, -9)$ and level 1. The non-critical slope bounds here are $v_p(U^\circ_{p,1}), v_p(U^\circ_{p,3}) < 4$, and $v_p(U^\circ_{p,2}) < 13$. At $p = 3$, there exists a Q-refinement $\tilde{\pi} = (\pi, \alpha)$ of π_3 with $v_3(\alpha) = 5 < 13$; this is non-Q-critical slope, so our constructions give a 3-adic L-function $\mathcal{L}_3(\tilde{\pi})$, uniquely determined by growth and interpolation. For each of the four Iwahori refinements $\tilde{\pi}' = (\pi, \alpha_1, \alpha_2, \alpha_3)$ above $\tilde{\pi}$, we have $v_p(\alpha_1), v_p(\alpha_3) \geqslant 5 \geqslant 4$, so each $\tilde{\pi}'$ has critical slope, and we have no unconditional construction of $\mathcal{L}_3(\tilde{\pi}')$ at Iwahori level.

From [66], we expect Theorem A is optimal, and one cannot improve the non-critical slope/growth bounds. We also expect that we construct *all* examples of p-adic L-functions that are: (a) attached to RASCARs π of $\mathrm{GL}_N(\mathbf{A})$ that are spherical and regular at p, and (b) uniquely determined by their interpolation/growth. An Iwahori approach does not give this.

A final, substantial, benefit is that at parahoric level, we can study local zeta integrals for parahoric-invariant vectors at $\mathfrak{p}|p$. For such vectors the zeta integrals with unramified twists have been computed in [13, §9], so we can prove our p-adic L-function has the expected interpolation at all characters.

1.2.4. Existence and uniqueness of Shalika families. Our second main result is the use of the p-adic L-functions of Theorem A to study the GL_{2n}-eigenvariety.

We now wish to vary λ, so let λ_π denote the weight of our (fixed) RASCAR π. This is a point in a rigid analytic *(parabolic) weight space* $\mathscr{W}^Q_{\lambda_\pi}$ of dimension $d + 1$ (see §3.1). For any (parahoric-at-p) level K (see (2.20)) and for any $h \geqslant h_p$ (defined in Theorem A), there exists:

– a slope-$\leqslant h$-adapted affinoid neighbourhood Ω of λ_π in $\mathscr{W}^Q_{\lambda_\pi}$,

– a sheaf $\mathscr{D}_\Omega$ on S_K interpolating $\mathscr{D}_\lambda$ as λ varies in Ω, and
– a local piece $\mathscr{E}_{\Omega,h}(K)$ of the global parabolic eigenvariety from [19], parametrising systems of Hecke eigenvalues occurring in the slope $\leqslant h$ part of $\mathrm{H}_c^t(S_K, \mathscr{D}_\Omega)$ with respect to U_p°, endowed with a finite weight map

$$w : \mathscr{E}_{\Omega,h}(K) \to \Omega.$$

We introduce some necessary terminology:

– A point $y \in \mathscr{E}_{\Omega,h}(K)$ is *classical* if this eigensystem appears in π_y^K for some cohomological automorphic representation π_y.
– A *Shalika point* is a classical point such that π_y is a RASCAR (i.e. π_y is cuspidal and admits a Shalika model).
– A *classical family through* $\tilde{\pi}$ *of level* K is an irreducible component of $\mathscr{E}_{\Omega,h}(K)$, containing $x_{\tilde{\pi}}$, that contains a Zariski-dense set of classical points.
– A *Shalika family* is a classical family containing a Zariski-dense set of Shalika points.

In our earlier works [11, 17, 18], we developed methods for studying $\mathrm{H}_c^\bullet(S_K, \mathscr{D}_\Omega)$ as an $\mathcal{O}_\Omega$-module. Cuspidal cohomology contributes to a continuous range of degrees $\{dn^2, dn^2+1, ..., t\}$. As we work in top degree t, for appropriate $\tilde{\pi}$, these methods easily yield a (Shalika) point $x_{\tilde{\pi}}$ in $\mathscr{E}_{\Omega,h}(K)$. To study the geometry around this point, it is crucial to understand the $\mathcal{O}_\Omega$-torsion in $\mathrm{H}_c^t(S_K, \mathscr{D}_\Omega)$. However, previous methods controlled this torsion only in bottom degree dn^2. For $n > 1$, cuspidal cohomology is supported in multiple degrees; so existing methods say nothing about the local geometry around $x_{\tilde{\pi}}$, including the dimension of components through $x_{\tilde{\pi}}$. Indeed, such methods do not even rule out $x_{\tilde{\pi}}$ being an isolated point. It is thus a non-trivial question if there are any classical families, let alone Shalika families, containing $\tilde{\pi}$.

Let $K_1(\tilde{\pi}) \subset G(\mathbf{A}_f)$ be the open compact subgroup that is parahoric at p and Whittaker new level (for π) away from p (see (7.2)). Our second main result, proved in Theorem 7.6, describes precisely the local geometry of $\mathscr{E}_{\Omega,h}(K_1(\tilde{\pi}))$ at $x_{\tilde{\pi}}$ and, in particular, answers positively the above question for RASCARs under very mild technical assumptions.

Theorem B. *Let π be a RASCAR of $G(\mathbf{A})$, and $\tilde{\pi}$ a Shalika p-refinement. Suppose that*

(a) λ_π is regular,

(b) $\tilde{\pi}$ has non-Q-critical slope, and

(c) the p-adic Galois representation ρ_π attached to π is absolutely irreducible.

Then $\mathscr{E}_{\Omega,h}(K_1(\tilde{\pi}))$ is étale over Ω at $x_{\tilde{\pi}}$. Up to shrinking Ω, w induces an isomorphism $\mathscr{C} \xrightarrow{\sim} \Omega$, where $\mathscr{C}$ is the connected component of $\mathscr{E}_{\Omega,h}(K_1(\tilde{\pi}))$ through $x_{\tilde{\pi}}$, and $\mathscr{C}$ is a Shalika family.

The same conclusions hold replacing (a) and (b) with the strictly weaker assumptions that

(a′) λ_π is H-regular (Definition 7.5) and $\mathcal{L}_p(\tilde{\pi})$ is non-zero, and

(b′) $\tilde{\pi}$ is strongly non-Q-critical (Definition 3.14).

See §7.1 for more details.

A technically challenging strategy for proving existence of the Shalika family $\mathscr{C}$ is to exhibit it as a p-adic Langlands transfer from GSpin_{2n+1}/F. This requires a number of additional hypotheses for general spin groups. Instead, we argue directly using the groups $\mathrm{H}_c^t(S_K, \mathscr{D}_\Omega)$ – without recourse to GSpin_{2n+1} – and prove existence via a novel application of our evaluation maps, suggested to us by Eric Urban. We briefly summarise this argument. We complete the construction of Figure 1, including the map $\mathrm{Ev}_\Omega : \mathrm{H}_c^t(S_K, \mathscr{D}_\Omega) \longrightarrow \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$, in §6. A standard argument provides a class $\Phi \in \mathrm{H}_c^t(S_K, \mathscr{D}_\Omega)$ lifting $\Phi_{\tilde{\pi}}$ under the natural specialisation map. Then:

- If $\mathcal{L}_p(\tilde{\pi}) \neq 0$, then – by the proof of Theorem A – we know $\mathrm{Ev}_\lambda(\Phi_{\tilde{\pi}}) \neq 0$. Hence, via the commutativity of the top square in Figure 1, we deduce $\mathrm{Ev}_\Omega(\Phi) \neq 0$.
- The map Ev_Ω is $\mathcal{O}_\Omega$-linear, and valued in the torsion-free $\mathcal{O}_\Omega$-module $\mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$. As $\mathrm{Ev}_\Omega(\Phi) \neq 0$, we deduce Φ is non-$\mathcal{O}_\Omega$-torsion.
- It follows that $\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)$ is a faithful $\mathcal{O}_\Omega$-module. We exploit this to deduce existence of a component in the eigenvariety of maximal dimension through $\tilde{\pi}$.

If there is a non-zero Deligne-critical L-value for π (which always exists when λ_π is regular), then the p-adic L-function is non-zero. The above argument then yields a classical family in the eigenvariety at the (sufficiently small) level $K(\tilde{\pi})$ used in §1.2.2. To upgrade this to a Shalika family, we again exploit our evaluation maps, giving (via Proposition 5.15) a criterion for being Shalika that is open over the eigenvariety.

It remains to prove uniqueness and étaleness. However, to exploit non-vanishing L-values, we must work at level $K(\tilde{\pi})$. As this is inexplicit, it is difficult to further control the geometry of families of level $K(\tilde{\pi})$. We can obtain more control by working at new tame level, that is at level $K_1(\tilde{\pi})$. We perform a delicate level-switching argument – using the local Langlands correspondence and p-adic Langlands functoriality (see §7.5) – to transfer the family to level $K_1(\tilde{\pi})$, where we then complete the proof of Theorem B.

We conclude §7 with an application of Theorem B to the global geometry of the eigenvariety. In Theorem 7.26, we show that if $\tilde{\pi}$ is as in Theorem B, and $\mathscr{I}$ is the (unique) irreducible component of the global eigenvariety through $\tilde{\pi}$, then every non-Q-critical slope classical point of $\mathscr{I}$ is Shalika. We actually prove more: that every point (classical or not) of $\mathscr{I}$ is symplectic, arising from GSpin_{2n+1}. Our proof goes through p-adic Langlands functoriality and occupies all of §7.7. We thank the referee for pushing us to prove such a result.

Remark 1.3. Theorem B describes the geometry of the Q-parabolic eigenvariety. This is natural in light of §1.2.3. Eigenvarieties for non-minimal parabolics have been well-studied; for a summary of constructions and arithmetic applications, see [19, Intro]. Major recent applications include Bloch–Kato for GSp_4 [60, §17] and modularity of elliptic curves over imaginary quadratic fields [30, §2.2].

The most traditional flavour of eigenvariety comes attached to a minimal parabolic subgroup, the Borel subgroup $B \subset G$, corresponding to Iwahoric level at p. With appropriate adaptation, and stronger assumptions, our methods also apply to this case, and we can prove the analogue of Theorem B for the Iwahori eigenvariety over the pure weight space (whose dimension grows with n). This – and applications to families of p-adic L-functions – is the subject of a follow-up paper [13] with Graham and Jorza.

1.2.5. p-adic L-functions in Shalika families. To prove Theorem A, we worked at a specific (inexplicit) level $K(\tilde{\pi})$, at which we have a precise connection to L-values. In Theorem B, we worked at a second specific (explicit) level $K_1(\tilde{\pi})$, where we obtain control over p-adic families.

When π is everywhere spherical away from p – that is, when π has tame level 1 – these two levels coincide. More precisely, we may take

$$K(\tilde{\pi}) = K_1(\tilde{\pi}) = \prod_{\mathfrak{p}|p} J_{\mathfrak{p}} \prod_{v\nmid p\infty} \mathrm{GL}_{2n}(\mathcal{O}_v).$$

In Chapter 8, we crucially exploit this to vary p-adic L-functions in families in the tame level 1 case (see Theorem C below).

The obstruction to generalising to higher tame level arises from local representation theory: namely, given a place $v \nmid p\infty$ such that π_v is ramified, we need to find explicit 'test vectors' in the Shalika model of π_v such that an attached Friedberg–Jacquet zeta integral computes the L-factor of π_v (see (2.15)). It is known that such vectors always exist abstractly, but explicit vectors – of the kind required for variation of p-adic L-functions – have not yet been found.

In §8.1 and §8.2, we describe, in very general terms, what kind of results would allow us to generalise Theorem C to higher tame level. On a concrete level, we hypothesise a possible theory of explicit test vectors, via *Shalika new vectors*, a Shalika analogue of the classical (Whittaker) newform theory of [51] (see Definition 8.2). Ramified examples where these hypotheses are satisfied have been found in the work [39] of the second author and Jorza.

We assume π satisfies this hypothesis for the rest of the paper. This allows us to work solely at a 'Shalika' level where we precisely see L-values, whilst still controlling the geometry of a modified eigenvariety $\mathscr{E}^{S}_{\Omega,h}$, defined by introducing additional diamond operators at places $v \in S = \{v \nmid p\infty : \pi_v \text{ ramified}\}$. In particular, we prove the following refinement of Theorem B.

Theorem B′. (Theorem 8.11). *Suppose that: (a) λ_π is regular, (b) $\tilde\pi$ has non-Q-critical slope, and (c) for all v, Hypothesis 8.6 holds for $c = c(\pi_v)$. Then:*

- *$\mathscr{E}^{S}_{\Omega,h}$ is étale over Ω at $\tilde\pi$, and (up to shrinking Ω) the connected component $\mathscr{C}$ through $\tilde\pi$ is a Shalika family mapping isomorphically onto Ω under w.*
- *$\mathscr{C}$ contains a very Zariski-dense set $\mathscr{C}_{\mathrm{nc}}$ of classical points satisfying the conditions of Definition 1.1 (see also Conditions 2.8). For all v, every point in $\mathscr{C}_{\mathrm{nc}}$ has a Shalika new vector of conductor $c(\pi_v)$.*
- *There exists an eigenclass $\Phi_{\mathscr{C}} \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)}, \mathscr{D}_\Omega)$, interpolating the classes $\Phi_{\tilde\pi_y}$ for $y \in \mathscr{C}_{\mathrm{nc}}$ (upto scaling by p-adic periods).*

When π has tame level 1, condition (c) is automatically satisfied with each $c(\pi_v) = 0$, and the eigenvariety $\mathscr{E}^{S}_{\Omega,h}$ is nothing but $\mathscr{E}_{\Omega,h}$ from above; so there are a ready supply of RASCARs where this result is unconditional. In general, we may also weaken assumptions as in Theorem B.

Given Theorem B′, standard methods give the analytic variation of $\mathcal{L}_p(\tilde\pi)$ over $\mathscr{C}$ as a formal consequence of our evaluation maps in families. The definition of the multi-variable p-adic L-function is summarised in row (T) of Figure 1.

Definition 1.4. Under the hypotheses of Theorem B′, let $\mathscr{C}$ be the Shalika family through $\tilde\pi$. Define the *p-adic L-function over $\mathscr{C}$* to be

$$\mathcal{L}_p^{\mathscr{C}} := \mathrm{Ev}_\Omega(\Phi_{\mathscr{C}}) \in \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega).$$

Let $\mathscr{X}(\mathrm{Gal}_p)$ be the $\mathbf{Q}_p$-rigid space of characters on Gal_p; then via the Amice transform ([23, Def. 5.1.5], building on [1, 73]), we may view $\mathcal{L}_p^{\mathscr{C}}$ as a rigid function

$$\mathcal{L}_p^{\mathscr{C}} : \mathscr{C} \times \mathscr{X}(\mathrm{Gal}_p) \to \mathbf{C}_p.$$

Our third main result (Theorem 8.21) is that $\mathcal{L}_p^{\mathscr{C}}$ interpolates $\mathcal{L}_p(\tilde\pi_y)$ as y varies in the set $\mathscr{C}_{\mathrm{nc}}$.

Theorem C. *Suppose the hypotheses of Theorem B′. Then at every $y \in \mathscr{C}_{\mathrm{nc}}$, there exists a set $\{c_y^\epsilon \in L^\times : \epsilon \in \{\pm 1\}^\Sigma\}$ of p-adic periods such that for every $\chi \in \mathscr{X}(\mathrm{Gal}_p)$, we have*

$$\mathcal{L}_p^{\mathscr{C}}(y,\chi) = c_y^{(\chi\eta)_\infty} \cdot \mathcal{L}_p(\tilde\pi_y, \chi). \tag{1.2}$$

Let $\mathscr{X}(\mathrm{Gal}_p^{\mathrm{cyc}}) \subset \mathscr{X}(\mathrm{Gal}_p)$ be the cyclotomic line, i.e. the Zariski-closure of $\{\chi_{\mathrm{cyc}}^j : j \in \mathbf{Z}\}$. Via Theorem A, $\mathcal{L}_p^{\mathscr{C}}$ simultaneously interpolates the values $L(\pi_y \times \chi, j + \frac{1}{2})$ over the set of points

$$\mathrm{Crit}(\mathscr{C}) = \Big\{(y, \chi\chi_{\mathrm{cyc}}^j) \in \mathscr{C} \times \mathscr{X}(\mathrm{Gal}_p^{\mathrm{cyc}}) : y \in \mathscr{C}_{\mathrm{nc}}, j \in \mathrm{Crit}(w(y)), \chi \text{ finite order}\Big\}.$$

The set $\mathrm{Crit}(\mathscr{C})$ is Zariski-dense in $\mathscr{C} \times \mathscr{X}(\mathrm{Gal}_p^{\mathrm{cyc}})$, so the restriction $\mathscr{L}_p^{\mathscr{C}}|_{\mathscr{C}\times\mathscr{X}(\mathrm{Gal}_p^{\mathrm{cyc}})}$ is uniquely determined by this interpolation. Specialising at $\tilde\pi$, we deduce:

Corollary 1.5. *Assume the hypotheses of Theorem B′. Up to a non-zero scalar, the restriction of $\mathcal{L}_p(\tilde\pi)$ to the cyclotomic line is uniquely determined by interpolation over* $\mathrm{Crit}(\mathscr{C})$.

If further Leopoldt's conjecture holds for F at p, then we obtain a similar uniqueness statement for $\mathcal{L}_p(\tilde\pi)$ itself, made precise in §8.5. These results should be compared to Theorem A, where we showed $\mathcal{L}_p(\tilde\pi)$ is determined by growth and interpolation, but only when $h_p < \#\mathrm{Crit}(\lambda_\pi)$.

Finally, let us highlight some examples for which the assumptions of Theorem C are satisfied. Let f be a classical cuspidal Hilbert eigenform of level 1 and weights $\geqslant 3$. The symmetric cube $\mathrm{Sym}^3(f)$ is a RASCAR for GL_4 of level 1 [45, Prop. 8.1.1]. When $\mathrm{Sym}^3(f)$ is non-Q-critical (e.g. if f itself is p-ordinary), then Theorem C shows that its p-adic L-function, as constructed in Theorem A, can be interpolated over the Hilbert cuspidal eigenvariety from [3]. More generally, Newton–Thorne recently showed that arbitrary symmetric powers of f are RACARs in [64, 63], and the odd symmetric powers are RASCARs.

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2. Automorphic preliminaries

The following fixes notation and recalls how to attach a compactly supported cohomology class (with p-adic coefficients) to a suitable automorphic representation. Everything here is standard.

2.1. Notation. Let F be a totally real number field of degree d over $\mathbf{Q}$, let $\mathcal{O}_F$ be its ring of integers and Σ the set of its real embeddings. Let $\mathbf{A} = \mathbf{A}_f \times \mathbf{R}$ denote the ring of adeles of $\mathbf{Q}$. For v a non-archimedean place of F, we let F_v be the completion of F at v, denote by $\mathcal{O}_v$ the ring of integers in F_v, and fix a uniformiser ϖ_v.

Let $n \geqslant 1$ and let G be the algebraic group $\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\mathrm{GL}_{2n}$, $B = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}B_{2n}$ be the Borel subgroup of upper triangular matrices, with opposite B^-, N and N^- be the unipotent radicals of B and B^- respectively, and $T = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}T_{2n}$ be the maximal split torus of diagonal matrices. We have decompositions $B = TN$ and $B^- = N^-T$. We let $K_\infty = C_\infty Z_G(\mathbf{R})$, where $C_\infty = O_{2n}(\mathbf{R})^d$ is the standard maximal compact subgroup of $G(\mathbf{R})$ and Z_G is the centre of G. For any reductive real Lie group A we let A° denote the connected component of the identity.

Let H denote the algebraic group $\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}(\mathrm{GL}_n \times \mathrm{GL}_n)$, which we frequently identify with its image under the natural embedding $\iota : H \hookrightarrow G$ given by $(h, h') \mapsto \left(\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right)$.

Let Z_H be the centre of H. We write $Q = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\left(\begin{smallmatrix} \mathrm{GL}_n & \mathrm{M}_n \\ 0 & \mathrm{GL}_n \end{smallmatrix}\right)$ for the maximal standard parabolic subgroup of G (containing B) whose Levi subgroup is H, and we denote by N_Q its unipotent radical.

Fix a rational prime p and an embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. We fix an extension of i_p to an isomorphism $i_p : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$. For each embedding $\sigma : F \hookrightarrow \mathbf{R}$ in Σ, there exists a unique prime

$\mathfrak{p}|p$ in F such that σ extends to an embedding $F_{\mathfrak{p}} \hookrightarrow \overline{\mathbf{Q}}_p$; we write $\mathfrak{p}(\sigma)$ for this prime, and let

$$\Sigma(\mathfrak{p}) \coloneqq \{\sigma \in \Sigma : \mathfrak{p}(\sigma) = \mathfrak{p}\}.$$

We let $\mathcal{O}_{F,p} \coloneqq \mathcal{O}_F \otimes \mathbf{Z}_p$.

Let $F^{p\infty}$ be the maximal abelian extension of F unramified outside $p\infty$, and let $\mathrm{Gal}_p \coloneqq \mathrm{Gal}(F^{p\infty}/F)$, which has the structure of a p-adic Lie group. Let $\mathrm{Gal}_p^{\mathrm{cyc}} \coloneqq \mathrm{Gal}(\mathbf{Q}^{p\infty}F/F)$.

Given an ideal $I \subset \mathcal{O}_F$ we let $\mathscr{U}(I) \coloneqq \{x \in \widehat{\mathcal{O}}_F^\times : x \equiv 1 \,(\mathrm{mod}\, I)\}$, and consider the narrow ray class group $\mathscr{C}\ell_F^+(I) = F^\times\backslash\mathbf{A}_F^\times/\mathscr{U}(I)F_\infty^{\times\circ}$.

All our group actions will be on the left. If M is a R-module, with a left action of a group Γ, then we write $M^\vee = \mathrm{Hom}_R(M,R)$, with associated left dual action

$$(\gamma\cdot\mu)(m) = \mu(\gamma^{-1}\cdot m).$$

For an affinoid rigid space X, we write $\mathcal{O}_X$ (or, for clarity of notation, occasionally $\mathcal{O}(X)$) for the ring of rigid functions on X, so $X = \mathrm{Sp}(\mathcal{O}_X)$.

2.2. The weights. Let $X^*(T)$ be the set of algebraic characters of T. Each element of $X^*(T)$ corresponds to an integral weight $\lambda = (\lambda_\sigma)_{\sigma\in\Sigma}$, where

$$\lambda_\sigma = (\lambda_{\sigma,1},\dots,\lambda_{\sigma,2n}) \in \mathbf{Z}^{2n}.$$

- We say that λ is *B-dominant* if it satisfies
$$\lambda_{\sigma,1} \geqslant \dots \geqslant \lambda_{\sigma,2n} \qquad \text{for each } \sigma\in\Sigma,$$
and we let $X_+^*(T) \subset X^*(T)$ be the subset of B-dominant weights.
- We say that λ is *pure* if there exists $\mathsf{w}\in\mathbf{Z}$, the *purity weight* of λ, such that
$$\lambda_{\sigma,i} + \lambda_{\sigma,2n-i+1} = \mathsf{w} \qquad \text{for each } \sigma\in\Sigma \text{ and } i\in\{1,\dots,2n\}.$$
We denote by $X_0^*(T) \subset X_+^*(T)$ the subset of pure B-dominant integral weights; these are exactly those supporting cuspidal cohomology [33, Lem. 4.9].
- We say λ is *regular* if
$$\lambda_{\sigma,i} > \lambda_{\sigma,i+1} \qquad \text{for all } \sigma \text{ and } i.$$
We emphasise that a RACAR π does not necessarily have regular weight; e.g. π can have (non-regular) weight $\lambda = (0,...,0)$.

For $\lambda\in X_+^*(T)$, we let V_λ be the algebraic irreducible representation of G of highest weight λ; for a sufficiently large field $L/\mathbf{Q}_p$, the L-points $V_\lambda(L)$ can be explicitly realised as

$$V_\lambda(L) = \{f : G(\mathbf{Q}_p)\to L \text{ algebraic} : \\ f(n^-tg) = \lambda(t)f(g) \text{ for all } n^-\in N^-(\mathbf{Q}_p), t\in T(\mathbf{Q}_p), g\in G(\mathbf{Q}_p)\}. \tag{2.1}$$

The (left) action of $G(\mathbf{Q}_p)$ is by right translation, i.e. $(h\cdot f)(g) = f(gh)$ for $g,h\in G(\mathbf{Q}_p)$ and $f\in V_\lambda$. Let $V_\lambda^\vee$ denote the linear dual, with its (left) dual action; we have an isomorphism $V_\lambda^\vee \cong V_{\lambda^\vee}$ where

$$\lambda^\vee = (\lambda_\sigma^\vee)_{\sigma\in\Sigma}, \qquad \lambda_\sigma^\vee = (-\lambda_{\sigma,2n},\dots,-\lambda_{\sigma,1}).$$

Note $\lambda^\vee = -w_{2n}(\lambda)$ is the contragredient of λ, for w_{2n} the longest Weyl element for G. Note the central characters of $V_\lambda^\vee$ and V_λ are inverse to each other, and if λ is pure, then as G-modules we have (e.g. [45, §2.3])

$$V_\lambda^\vee \cong V_\lambda\otimes[\mathrm{N}_{F/\mathbf{Q}}\circ\det]^{-\mathsf{w}}.$$

By Zariski-density any $f \in V_\lambda$ is uniquely determined by $f|_{G(\mathbf{Z}_p)}$. We have a natural integral subspace $V_\lambda(\mathcal{O}_L)$ of $f \in V_\lambda(L)$ such that $f(G(\mathbf{Z}_p)) \subset \mathcal{O}_L$; we let $V_\lambda^\vee(\mathcal{O}_L) = \mathrm{Hom}_{\mathcal{O}_L}(V_\lambda(\mathcal{O}_L), \mathcal{O}_L)$.

Let $X^*(H)$ be the set of algebraic characters of H. Each element of $X^*(H)$ is identified with an integral weight

$$(j, j') = (j_\sigma, j'_\sigma)_{\sigma\in\Sigma}, \qquad j_\sigma, j'_\sigma \in \mathbf{Z}.$$

We say $(j, j') \in X^*(H)$ is *Q-dominant* if $j_\sigma \geqslant j'_\sigma$ for each $\sigma \in \Sigma$, and let $X^*_+(H) \subset X^*(H)$ be the subset of Q-dominant weights. We say that $(j, j') \in X^*(H)$ is *pure* if there exists $\mathsf{w} \in \mathbf{Z}$ such that $j_\sigma + j'_\sigma = \mathsf{w}$ for all $\sigma \in \Sigma$, and let

$$X^*_0(H) \subset X^*_+(H)$$

be the subset of pure Q-dominant weights. Since $B \subset Q$, we naturally have

$$X^*(H) \subset X^*(T), \qquad X^*_+(H) \subset X^*_+(T), \qquad X^*_0(H) \subset X^*_0(T).$$

Given a pure B-dominant integral weight $\lambda = (\lambda_\sigma)_{\sigma\in\Sigma}$, we define a set

$$\mathrm{Crit}(\lambda) \coloneqq \{j \in \mathbf{Z} : -\lambda_{\sigma,n} \leqslant j \leqslant -\lambda_{\sigma,n+1}\ \forall \sigma \in \Sigma\}. \tag{2.2}$$

If π is a RACAR for $G(\mathbf{A})$ of weight λ (which we take to mean cohomological with respect to $V_\lambda^\vee$, in the sense of §2.5), then [45, §6.1] proves

$j \in \mathrm{Crit}(\lambda) \iff$ for all finite order Hecke characters χ of F, the L-value $L(\pi \otimes \chi, j + \frac{1}{2})$ is critical in the sense of Deligne.

2.3. Local systems and Betti cohomology. Let $K \subset G(\mathbf{A}_f)$ be an open compact subgroup. The *locally symmetric space of level K* is the $d(2n-1)(n+1)$-dimensional real orbifold

$$S_K \coloneqq G(\mathbf{Q})\backslash G(\mathbf{A})/KK_\infty^\circ. \tag{2.3}$$

2.3.1. Archimedean local systems. Let M be a left $G(\mathbf{Q})$-module such that $Z_G(\mathbf{Q}) \cap KK_\infty^\circ$ acts trivially (else, the local systems we define are zero). To M we attach a local system $\mathcal{M} = \mathcal{M}_K$ on S_K, defined as the locally constant sections of

$$G(\mathbf{Q})\backslash[G(\mathbf{A}) \times M]/KK_\infty^\circ \longrightarrow S_K,$$

with action $\gamma(g, m)kz = (\gamma gkz, \gamma \cdot m)$. We use calligraphic letters for such local systems.

Applying to $M = V_\lambda^\vee(E)$ for a characteristic zero field E, we can consider Betti cohomology groups $\mathrm{H}^\bullet_*(S_K, \mathcal{V}^\vee_{\lambda,K}(E))$ where $* = \varnothing$ for the usual and $* = c$ for the compactly supported ones. Given any finite index subgroup $K' \subset K$ one has

$$p^*_{K',K}\mathcal{V}^\vee_{\lambda,K} = \mathcal{V}^\vee_{\lambda,K'},$$

where $p_{K',K} : S_{K'} \to S_K$ is the natural projection. The adjunction yields then a natural homomorphism

$$\mathrm{H}^\bullet_*(S_K, \mathcal{V}^\vee_{\lambda,K}(E)) \to \mathrm{H}^\bullet_*(S_K, (p_{K',K})_* p^*_{K',K}\mathcal{V}^\vee_{\lambda,K}(E)) = \mathrm{H}^\bullet_*(S_{K'}, \mathcal{V}^\vee_{\lambda,K'}(E)).$$

allowing us to consider the 'infinite level' cohomology

$$\mathrm{H}^\bullet_*(S^G, \mathcal{V}^\vee_\lambda(E)) \coloneqq \varinjlim_K \mathrm{H}^\bullet_*(S_K, \mathcal{V}^\vee_{\lambda,K}(E)).$$

This admits a natural $G(\mathbf{A}_f)$-action, whose K-invariants are $\mathrm{H}^\bullet_*(S_K, \mathcal{V}^\vee_{\lambda,K}(E))$. Because of this compatibility, for ease of notation we henceforth drop the subscript K and write only $\mathcal{V}^\vee_\lambda(E)$.

2.3.2. Non-archimedean local systems. For the rest of the paper, we will work with cohomology with p-adic coefficients, for which we need an appropriate notion of non-archimedean local systems on S_K. Let M be a left K-module on which the centre $Z_G(\mathbf{Q}) \cap KK_\infty^\circ$ acts trivially. To M, we attach a local system $\mathscr{M} = \mathscr{M}_K$ on S_K as the locally constant sections of

$$G(\mathbf{Q})\backslash[G(\mathbf{A}) \times M]/KK_\infty^\circ \longrightarrow S_K,$$

with action $\gamma(g,m)kz = (\gamma gkz, k^{-1}\cdot m)$. For these we use script letters, e.g. $\mathscr{V}, \mathscr{D}$.

Suppose now M has a left action of $G(\mathbf{A}_f)$. This gives left actions of $G(\mathbf{Q})$ and K on M, and we get associated (archimedean and non-archimedean) local systems $\mathcal{M}$ and $\mathscr{M}$ attached to M. One may check (see [81, §1.2.2]) there is an isomorphism

$$\mathcal{M} \cong \mathscr{M}, \qquad \text{given on sections by } (g,m) \mapsto (g, g_f^{-1}\cdot m).$$

The following is the example of most importance to us. If $L/\mathbf{Q}_p$ contains the field of definition of λ, then $M = V_\lambda(L)$ can be realised as a space of functions $f : G(\mathbf{Q}_p) \to L$. If $\mu \in V_\lambda^\vee(L)$, $f \in V_\lambda(L)$, and $g \in G(\mathbf{Q}_p)$, then $V_\lambda^\vee(L)$ carries an action of $h \in G(\mathbf{A}_f)$ by

$$(h\cdot\mu)[f(g)] := \mu[f(gh_p^{-1})],$$

where h_p is the image of h under the projection $G(\mathbf{A}_f) \to G(\mathbf{Q}_p)$. We get two local systems $\mathcal{V}_\lambda^\vee(L)$ and $\mathscr{V}_\lambda^\vee(L)$, and as above, we get an isomorphism $\mathcal{V}_\lambda^\vee \cong \mathscr{V}_\lambda^\vee$.

2.3.3. Hecke operators. Let M be a left module for $G(\mathbf{Q})$ (resp. K), and let $\gamma \in G(\mathbf{A}_f)$, which we suppose acts on M. As in [38, §1.4] define a Hecke operator on $\mathrm{H}_c^\bullet(S_K, \mathcal{M})$ by

$$[K\gamma K] := \mathrm{Tr}(p_{\gamma K\gamma^{-1}\cap K,K}) \circ [\gamma] \circ p^*_{K\cap\gamma^{-1}K\gamma,K} : \mathrm{H}_c^\bullet(S_K,\mathcal{M}) \to \mathrm{H}_c^\bullet(S_K,\mathcal{M}),$$

where Tr is the trace map attached to the finite cover $S_{\gamma K\gamma^{-1}\cap K} \to S_K$, $p_{K',K} : S_{K'} \to S_K$ is the natural projection, and

$$[\gamma] : \mathrm{H}_c^\bullet(S_{K\cap\gamma^{-1}K\gamma},\mathcal{M}) \to \mathrm{H}_c^\bullet(S_{\gamma K\gamma^{-1}\cap K},\mathcal{M})$$

is given on local systems by $(g,m) \mapsto (g\gamma^{-1}, \gamma\cdot m)$ (and similarly for $\mathscr{M}$-coefficients).

One can check that if M is a $G(\mathbf{A}_f)$-module as in §2.3.2, then the isomorphism

$$\mathrm{H}_c^\bullet(S_K,\mathcal{M}) \xrightarrow{\sim} \mathrm{H}_c^\bullet(S_K,\mathscr{M}) \tag{2.4}$$

induced from the isomorphism $\mathcal{M} \cong \mathscr{M}$ is Hecke-equivariant [81, §1.2.5].

2.3.4. Operators at infinity. If $\sigma \in \Sigma$, then $K_\sigma/K_\sigma^\circ = \{\pm1\}$, and thus $K_\infty/K_\infty^\circ = \{\pm1\}^\Sigma$. Any character $\epsilon : K_\infty/K_\infty^\circ \to \{\pm1\}$ can also be identified with an element of $\{\pm1\}^\Sigma$. If M is a module upon which K_∞/K_∞° acts, let M^ϵ be the submodule upon which the action is by ϵ. If M is a vector space over a field of characteristic $\neq 2$, then $M = \oplus_\epsilon M^\epsilon$. Since the group acts naturally on S_K and its cohomology, and this action commutes with the $G(\mathbf{A}_f)$-action, we thus obtain decompositions of its cohomology into Hecke-stable submodules (see e.g. [45, p.15]).

2.4. The spherical Hecke algebra. Let π be a RACAR of $G(\mathbf{A})$ of weight λ, and let

$$S = \{v \nmid p\infty : \pi_v \text{ not spherical}\}$$

be the set of bad places for π. Let

$$K = \textstyle\prod_{v\nmid\infty} K_v \subset G(\mathbf{A}_f)$$

be an open compact subgroup such that $\pi_f^K \neq 0$; for $v \notin S \cup \{\mathfrak{p}|p\}$, we take

$$K_v = K_v^\circ := \mathrm{GL}_{2n}(\mathcal{O}_v).$$

We now introduce the (unramified) Hecke algebra. Let $X_*^+(T_{2n})$ denote the set of algebraic B-dominant cocharacters of $T_{2n} \subset \mathrm{GL}_{2n}$, identified with tuples $\nu = (\nu_1, \dots, \nu_{2n}) \in \mathbf{Z}^{2n}$ with

$$\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_{2n}, \qquad \text{via} \qquad x \mapsto \mathrm{diag}(x^{\nu_1}, \dots, x^{\nu_{2n}}).$$

Definition 2.1. For $v \notin S \cup \{\mathfrak{p}|p\}$, and any $\nu \in X_*^+(T_{2n})$, let $T_{\nu,v} := [K_v^\circ \nu(\varpi_v) K_v^\circ]$. The *unramified Hecke algebra of level K* is the commutative algebra $\mathcal{H}'$ generated by all such operators:

$$\mathcal{H}' := \mathbf{Z}[T_{\nu,v} : \nu \in X_*^+(T_{2n}), v \notin S \cup \{\mathfrak{p}|p\}].$$

For any choice of K such that $K_v = K_v^\circ$ for $v \notin S \cup \{\mathfrak{p}|p\}$, the algebra $\mathcal{H}'$ acts on π^K via right translation, and on $\mathrm{H}^\bullet_*(S_K, -)$ as described in §2.3.3.

Definition 2.2. Let E be a number field containing the Hecke field of π_f. Attached to π we have a homomorphism

$$\psi_\pi : \mathcal{H}' \otimes E \to E$$

which for $\nu \in X_*^+(T_{2n})$ and $v \notin S \cup \{\mathfrak{p}|p\}$ sends $T_{\nu,v}$ to its eigenvalue acting on the line $\pi_v^{K_v^\circ}$. Let $\mathfrak{m}_\pi := \ker(\psi_\pi)$, a maximal ideal in $\mathcal{H}' \otimes E$. If L is any field containing E, we get an induced maximal ideal in $\mathcal{H}' \otimes L$, which in an abuse of notation we also denote $\mathfrak{m}_\pi$.

Note in the set-up above, if M is a finite-dimensional L-vector space with an action of $\mathcal{H}'$, then the localisation $M_{\mathfrak{m}_\pi}$ is the generalised eigenspace $M[\![\mathfrak{m}_\pi]\!]$ attached to ψ_π.

2.5. Cohomology classes attached to RACARs. We now attach compactly supported cohomology classes to RACARs. All the discussions in §2.5 are standard, and culminate in Proposition 2.3 below.

First, we recall standard results on cuspidal cohomology from [25, 33]. For a weight $\lambda \in X_0^*(T)$, the Betti cohomology $\mathrm{H}^\bullet(S^G, \mathcal{V}_\lambda^\vee(\mathbf{C}))$ is an admissible $G(\mathbf{A}_f)$-module. It admits a $G(\mathbf{A}_f)$-submodule $\mathrm{H}^\bullet_{\mathrm{cusp}}(S^G, \mathcal{V}_\lambda^\vee(\mathbf{C}))$, which we can describe using relative Lie algebra cohomology as

$$\mathrm{H}^\bullet_{\mathrm{cusp}}(S^G, \mathcal{V}_\lambda^\vee(\mathbf{C})) = \bigoplus_\pi \mathrm{H}^\bullet\big(\mathfrak{g}_\infty, K_\infty^\circ; \pi_\infty \otimes V_\lambda^\vee(\mathbf{C})\big) \otimes \pi_f, \tag{2.5}$$

where $\mathfrak{g}_\infty = \mathrm{Lie}(G_\infty)$ and the sum is over all RACARs π of $G(\mathbf{A})$. If π contributes non-trivially to the direct sum in (2.5), then we say it has *weight* λ, and it then contributes to all degrees i where

$$\mathrm{H}^i(\mathfrak{g}_\infty, K_\infty^\circ; \pi_\infty \otimes V_\lambda^\vee(\mathbf{C})) \neq 0,$$

which by [33, p. 120] (see also [45, (3.4.2)]) is for i in the range

$$dn^2 \leqslant i \leqslant d(n^2 + n - 1) =: t. \tag{2.6}$$

Denote the K-invariants by

$$\mathrm{H}^\bullet_{\mathrm{cusp}}(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C})) := \mathrm{H}^\bullet_{\mathrm{cusp}}(S^G, \mathcal{V}_\lambda^\vee(\mathbf{C}))^K \subset \mathrm{H}^\bullet(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C})).$$

If π has weight λ, then π contributes to $\mathrm{H}^\bullet_{\mathrm{cusp}}(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C}))$ if and only if $\pi_f^K \neq \{0\}$.

The action of the Hecke algebra $\mathcal{H}'$ on $\mathrm{H}^\bullet(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C}))$ preserves the cuspidal subspace. If we take K-invariants in (2.5), and localise the resulting $\mathcal{H}'$-module at $\mathfrak{m}_\pi \subset \mathcal{H}'$, then by the Strong Multiplicity One Theorem only the π-summand in the right-hand side survives, i.e.

$$\mathrm{H}^\bullet_{\mathrm{cusp}}(S^G, \mathcal{V}_\lambda^\vee(\mathbf{C}))_{\mathfrak{m}_\pi} = \mathrm{H}^\bullet\big(\mathfrak{g}_\infty, K_\infty^\circ; \pi_\infty \otimes V_\lambda^\vee(\mathbf{C})\big) \otimes \pi_f^K.$$

There is a natural action of K_∞/K_∞° on the factor at infinity, hence on cuspidal cohomology, and taking ϵ-parts for $\epsilon \in \{\pm 1\}^\Sigma$ (as in §2.3.4), we then obtain

$$\mathrm{H}^\bullet_{\mathrm{cusp}}(S^G, \mathcal{V}_\lambda^\vee(\mathbf{C}))^\epsilon_{\mathfrak{m}_\pi} = \mathrm{H}^\bullet\big(\mathfrak{g}_\infty, K_\infty^\circ; \pi_\infty \otimes V_\lambda^\vee(\mathbf{C})\big)^\epsilon \otimes \pi_f^K. \tag{2.7}$$

As in [45, §4.1], for degree t (that is, at the top of the range (2.6)) we have

$$\dim_{\mathbf{C}}\ \mathrm{H}^t(\mathfrak{g}_\infty, K_\infty^\circ; \pi_\infty \otimes V_\lambda^\vee(\mathbf{C}))^\epsilon = 1 \tag{2.8}$$

for all $\epsilon \in \{\pm 1\}^\Sigma$. Fixing a basis Ξ_∞^ϵ of (2.8) fixes an $\mathcal{H}'$-equivariant isomorphism

$$\pi_f^K \xrightarrow{\sim} \mathrm{H}^t_{\mathrm{cusp}}(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C}))^\epsilon_{\mathfrak{m}_\pi},$$

defined by $\varphi_f \mapsto \Xi_\infty^\epsilon \otimes \varphi_f$.

By [40], if π is a RACAR, then it does not contribute to the Eisenstein cohomology, so after localising we have

$$\mathrm{H}^\bullet_{\mathrm{cusp}}(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C}))_{\mathfrak{m}_\pi} \cong \mathrm{H}^\bullet(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C}))_{\mathfrak{m}_\pi}.$$

Since π is cuspidal, the boundary cohomology vanishes after localising at $\mathfrak{m}_\pi$; so the boundary exact sequence yields an isomorphism

$$\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C}))_{\mathfrak{m}_\pi} \xrightarrow{\sim} \mathrm{H}^\bullet(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C}))_{\mathfrak{m}_\pi}.$$

Combining gives an isomorphism

$$\pi_f^K \xrightarrow{\sim} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C}))^\epsilon_{\mathfrak{m}_\pi}. \tag{2.9}$$

Finally, via our fixed isomorphism $i_p : \mathbf{C} \cong \overline{\mathbf{Q}}_p$ and the isomorphism (2.4), we have isomorphisms

$$\begin{aligned}\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C})) &\xrightarrow[\sim]{i_p} \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathcal{V}_\lambda^\vee(\overline{\mathbf{Q}}_p)) \\ &\xrightarrow[\sim]{(2.4)} \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p)).\end{aligned} \tag{2.10}$$

As all the maps above are Hecke-equivariant, combining we finally deduce:

Proposition 2.3. *There is a Hecke-equivariant isomorphism*

$$\pi_f^K \xrightarrow{\sim} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))^\epsilon_{\mathfrak{m}_\pi}. \tag{2.11}$$

This isomorphism is non-canonical, depending on the choice of basis Ξ_∞^ϵ *of* (2.8).

2.6. Shalika models and Friedberg–Jacquet integrals. We recall some relevant facts about Shalika models (see e.g. [45, §1,§3.1]). Let

$$\mathcal{S}_{/F} = \{s = \left(\begin{smallmatrix} h & \\ & h\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} I_n & X \\ & I_n\end{smallmatrix}\right) : h \in \mathrm{GL}_n, X \in \mathrm{M}_n\}$$

be the Shalika subgroup of $\mathrm{GL}_{2n/F}$, and $\mathcal{S} = \mathrm{Res}_{F/\mathbf{Q}}\mathcal{S}_{/F}$. Let ψ be the standard non-trivial additive character of $F\backslash\mathbf{A}_F$ from [38, §4.1], and let η be a Hecke character of $F^\times\backslash\mathbf{A}_F^\times$. For $s \in \mathcal{S}$, write

$$(\eta \otimes \psi)(s) = \eta(\det(h))\psi(\mathrm{Tr}(X)).$$

A cuspidal automorphic representation π of $G(\mathbf{A})$ (of weight λ) is said to have an (η,ψ)-*Shalika model* if there exist $\varphi \in \pi$ and $g \in G(\mathbf{A})$ such that

$$\mathcal{S}_\psi^\eta(\varphi)(g) := \int_{Z_G(\mathbf{A})\mathcal{S}(\mathbf{Q})\backslash\mathcal{S}(\mathbf{A})} \varphi(sg)\ (\eta \otimes \psi)^{-1}(s)ds \neq 0. \tag{2.12}$$

This forces η^n to be equal to the central character of π, and hence $\eta = \eta_0|\cdot|^{\mathsf{w}}$, where η_0 has finite order and w is the purity weight of λ. If (2.12) holds, then $\mathcal{S}_\psi^\eta$ defines an intertwining $\pi \hookrightarrow \mathrm{Ind}_{\mathcal{S}(\mathbf{A})}^{G(\mathbf{A})}(\eta\otimes\psi)$, realising π inside the space of functions $W : G(\mathbf{A}) \to \mathbf{C}$ satisfying

$$W\left(\left(\begin{smallmatrix} h & 0\\ 0 & h\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & X\\ 0 & 1\end{smallmatrix}\right)\bullet\right) = \eta(\det(h))\psi(\mathrm{tr}(X))W(\bullet) \quad \forall h \in \mathrm{GL}_n(F), X \in M_n(F). \tag{2.13}$$

If π has an (η,ψ)-Shalika model, then for each place v of F the local component π_v has a local (η_v,ψ_v)-Shalika model [45, §3.2], that is, we have an intertwining

$$\mathcal{S}_{\psi_v}^{\eta_v} : \pi_v \hookrightarrow \mathrm{Ind}_{\mathcal{S}(F_v)}^{\mathrm{GL}_{2n}(F_v)}(\eta_v\otimes\psi_v). \tag{2.14}$$

Remark 2.4. Note (2.12) defines a canonical global intertwining. We emphasise that the local intertwinings are *not* canonical. However the local Shalika model is unique in the sense that

$$\dim_{\mathbf{C}}\ \mathrm{Hom}_{\mathrm{GL}_{2n}(F_v)}\left[\pi_v,\ \mathrm{Ind}_{\mathcal{S}(F_v)}^{\mathrm{GL}_{2n}(F_v)}(\eta_v\otimes\psi_v)\right] = 1$$

(see [65, 31]), so the image $\mathcal{S}_{\psi_v}^{\eta_v}(\pi_v)$ of $\mathcal{S}_{\psi_v}^{\eta_v}$ is canonical. We henceforth fix a (non-canonical) choice of intertwining $\mathcal{S}_{\psi_f}^{\eta_f}$ of π_f (or equivalently, via (2.12), an intertwining $\mathcal{S}_{\psi_\infty}^{\eta_\infty}$ of π_∞).

When π_v is spherical it is shown in [7, Prop. 1.3] that it admits a (η_v,ψ_v)-Shalika model if and only if $\pi_v^\vee = \pi_v\otimes\eta_v^{-1}$. In this case we deduce η_v is unramified.

Let π be a cuspidal automorphic representation of $G(\mathbf{A})$, and χ a finite order Hecke character for F. For $W \in \mathcal{S}_\psi^\eta(\pi)$ (the image of π under $\mathcal{S}_\psi^\eta$) consider the *Friedberg–Jacquet* zeta integral

$$\zeta(s,W,\chi) := \int_{\mathrm{GL}_n(\mathbf{A}_F)} W\left[\begin{pmatrix} h & \\ & I_n\end{pmatrix}\right]\ \chi(\det(h))\ |\det(h)|^{s-\frac{1}{2}}\ dh,$$

which converges absolutely in a right-half plane and extends to a meromorphic function in $s\in\mathbf{C}$. When $W = \otimes_v W_v$ for $W_v \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi_v)$, this integral is a product of local zeta integrals $\zeta(s,W_v,\chi_v)$.

A *Friedberg–Jacquet test vector* $W_v^{\mathrm{FJ}} \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi_v)$ is a vector such that for all unramified quasi-characters $\chi_v : F_v^\times \to \mathbf{C}^\times$, we have

$$\zeta_v\left(s+\tfrac{1}{2}, W_v^{\mathrm{FJ}},\chi_v\right) = [\mathrm{N}_{F/\mathbf{Q}}(v)^s\chi_v(\varpi_v)]^{n\delta_v}\cdot L\left(\pi_v\otimes\chi_v, s+\tfrac{1}{2}\right), \tag{2.15}$$

where δ_v is the valuation of the different of F_v and $L(\pi_v\otimes\chi_v, s+\frac{1}{2})$ is the Langlands L-function of $\pi_v\otimes\chi_v$. By [41, Prop. 3.1], if π is a RACAR admitting a (η,ψ)-Shalika model, then for every finite place v there exists such a Friedberg–Jacquet test vector in $\mathcal{S}_{\psi_v}^{\eta_v}(\pi_v)$. If π_v is spherical, then one can take W_v^{FJ} to be a spherical vector, i.e. a vector fixed by $\mathrm{GL}_{2n}(\mathcal{O}_v)$, normalised so that $W_v^{\mathrm{FJ}}(t_v^{-\delta_v}) = 1$ [41, Prop. 3.2], [38, Prop. 3.3].

2.7. Parahoric p-refinements. Let

$$J_{\mathfrak{p}} = \{g\in \mathrm{GL}_{2n}(\mathcal{O}_{\mathfrak{p}}) : g\ (\mathrm{mod}\ \mathfrak{p}) \in Q(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})\} \subset \mathrm{GL}_{2n}(F_{\mathfrak{p}}) \tag{2.16}$$

be the parahoric subgroup of type Q. We will always assume $\pi_{\mathfrak{p}}$ is Q-parahoric-spherical, that is, admits $J_{\mathfrak{p}}$-fixed vectors. Recall $\iota(h,h') = \left(\begin{smallmatrix} h & 0\\ 0 & h'\end{smallmatrix}\right)$, and let $t_{\mathfrak{p}} = \iota(\varpi_{\mathfrak{p}}I_n, I_n)$, recalling $\varpi_{\mathfrak{p}}$ is a uniformiser of $F_{\mathfrak{p}}$. On $\pi_{\mathfrak{p}}^{J_{\mathfrak{p}}}$, we have the Hecke operator

$$U_{\mathfrak{p}} := [J_{\mathfrak{p}}t_{\mathfrak{p}}J_{\mathfrak{p}}].$$

Definition 2.5. A *Q-refinement* $\tilde\pi_{\mathfrak{p}} = (\pi_{\mathfrak{p}},\alpha_{\mathfrak{p}})$ of $\pi_{\mathfrak{p}}$ is a choice of $U_{\mathfrak{p}}$-eigenvalue $\alpha_{\mathfrak{p}}$ on $\pi_{\mathfrak{p}}^{J_{\mathfrak{p}}}$. We say a Q-refinement $\tilde\pi_{\mathfrak{p}}$ is *regular* if $\alpha_{\mathfrak{p}}$ is a simple $U_{\mathfrak{p}}$-eigenvalue on $\pi_{\mathfrak{p}}^{J_{\mathfrak{p}}}$; that is,

$$\dim_{\mathbf{C}}\ \pi_{\mathfrak{p}}^{J_{\mathfrak{p}}}[\![U_{\mathfrak{p}}-\alpha_{\mathfrak{p}}]\!] = 1.$$

We say $\tilde\pi_{\mathfrak{p}}$ is *Shalika* if it is regular and if for any generator $W_{\mathfrak{p}}$ of $\mathcal{S}_{\psi_{\mathfrak{p}}}^{\eta_{\mathfrak{p}}}(\pi_{\mathfrak{p}}^{J_{\mathfrak{p}}})[\![U_{\mathfrak{p}}-\alpha_{\mathfrak{p}}]\!]$, we have

$$W_{\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}) \neq 0. \tag{2.17}$$

Remark 2.6. If $\tilde{\pi}_{\mathfrak{p}}$ is a Shalika Q-refinement, then for $h \in \mathrm{GL}_n(\mathcal{O}_{\mathfrak{p}})$ we see

$$\eta(\det(h)) \cdot W_{\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}) = W_{\mathfrak{p}}(\iota(h,h)t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}) = W_{\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}\iota(h,h)) = W_{\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}),$$

using (2.13) in the first equality and $J_{\mathfrak{p}}$-invariance in the last equality. By non-vanishing, we have $\eta_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}^{\times}) = 1$, so $\eta_{\mathfrak{p}}$ is unramified.

Condition (2.17) is motivated by non-vanishing of a local zeta integral; see Proposition 5.20. Indeed, if $W_{\mathfrak{p}} \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi_{\mathfrak{p}})^{J_{\mathfrak{p}}}$ is any vector, a relevant local twisted Friedberg–Jacquet zeta integral attached to $W_{\mathfrak{p}}$ is computed in [38, Prop. 3.4] and shown to be a scalar multiple of $W_{\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}})$ (see Lemma 5.20).

The following stronger assumptions give a ready source of $\tilde{\pi}_{\mathfrak{p}}$ as above. Suppose $\pi_{\mathfrak{p}}$ is spherical (admits $\mathrm{GL}_{2n}(\mathcal{O}_{\mathfrak{p}})$-fixed vectors). In this case $\pi_{\mathfrak{p}} = \mathrm{Ind}_B^G \theta_{\mathfrak{p}}$ is an unramified principal series, for $\theta_{\mathfrak{p}} = (\theta_{\mathfrak{p},1}, ..., \theta_{\mathfrak{p},2n})$ an unramified character of $T(F_{\mathfrak{p}})$. Ash–Ginzburg show in [7, Prop. 1.3] that such $\pi_{\mathfrak{p}}$ have an $(\eta_{\mathfrak{p}}, \psi_{\mathfrak{p}})$-Shalika model if and only if the $\theta_{\mathfrak{p},i}$'s can be ordered so that $\theta_{\mathfrak{p},i}\theta_{\mathfrak{p},n+i} = \eta_{\mathfrak{p}}$ for $1 \leqslant i \leqslant n$. Then [38, Lem. 3.6] shows our notion of Shalika is equivalent to being Q-regular in [38, Def. 3.5]:

Proposition 2.7. *[38, Lem. 3.6]. Suppose $\pi_{\mathfrak{p}} = \mathrm{Ind}_B^G \theta_{\mathfrak{p}}$ is spherical, that $\theta_{\mathfrak{p},i}\theta_{\mathfrak{p},n+i} = \eta_{\mathfrak{p}}$ for $1 \leqslant i \leqslant n$. Let $\alpha_{\mathfrak{p}} = q_{\mathfrak{p}}^{n^2/2}\theta_{\mathfrak{p},n+1}(\varpi_{\mathfrak{p}}) \cdots \theta_{\mathfrak{p},2n}(\varpi_{\mathfrak{p}})$, where $q_{\mathfrak{p}} = \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{p})$. Suppose $(\pi_{\mathfrak{p}}, \alpha_{\mathfrak{p}})$ is a regular Q-refinement. Then it is a Shalika Q-refinement.*

For spherical $\pi_{\mathfrak{p}}$, the Q-refinements that can be described as in Proposition 2.7 are exactly the *Q-spin* refinements from [14]. If all the $\binom{2n}{n}$ possible Q-refinements of $\pi_{\mathfrak{p}}$ are different, then 2^n of them are Q-spin, so this condition covers a wide range of $\tilde{\pi}_{\mathfrak{p}}$.

Globally, a *(Shalika) Q-refined RA(S)CAR* is a tuple $\tilde{\pi} = (\pi, \{\alpha_{\mathfrak{p}}\}_{\mathfrak{p}|p})$, for a RA(S)CAR π where $\pi_{\mathfrak{p}}$ is Q-parahoric-spherical and $(\pi_{\mathfrak{p}}, \alpha_{\mathfrak{p}})$ is a (Shalika) Q-refinement for each $\mathfrak{p}|p$. We will construct p-adic L-functions and families for any Shalika Q-refined RASCAR, i.e. assuming only that $\pi_{\mathfrak{p}}$ is parahoric-spherical at each $\mathfrak{p}|p$. However, if we assume further that each $\tilde{\pi}_{\mathfrak{p}}$ is as in Proposition 2.7, we obtain slightly stronger results.

2.8. Running conditions on $\tilde{\pi}$. We finally collect our running assumptions. Fix for the rest of the paper a finite order Hecke character η_0 of F. We work with two levels of generality; our results apply under (C2), but are more precise under the stronger assumption (C2′).

Conditions 2.8. Let π be a RACAR of $G(\mathbf{A})$ of weight λ such that

(C1) π admits a global $(\eta_0|\cdot|^{\mathsf{w}}, \psi)$-Shalika model, for w the purity weight of π;

(C2) for each $\mathfrak{p}|p$, $\pi_{\mathfrak{p}}$ is parahoric-spherical admitting a Shalika Q-refinement $\tilde{\pi}_{\mathfrak{p}} = (\pi_{\mathfrak{p}}, \alpha_{\mathfrak{p}})$, i.e.

$$\dim_{\mathbf{C}} \ \mathcal{S}_{\psi_{\mathfrak{p}}}^{\eta_{\mathfrak{p}}}(\pi_{\mathfrak{p}}^{J_{\mathfrak{p}}})[\![U_{\mathfrak{p}} - \alpha_{\mathfrak{p}}]\!] = 1 \tag{2.18}$$

(for $\eta_{\mathfrak{p}} = \eta_{0,\mathfrak{p}}|\cdot|_{\mathfrak{p}}^{\mathsf{w}}$) and this line admits a generator $W_{\mathfrak{p}}$ such that $W_{\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}) = 1$.

By Remark 2.6, (C2) forces $\eta_{\mathfrak{p}}$ to be unramified. We also use:

Conditions 2.8′. Let π be a RACAR of $G(\mathbf{A})$ of weight λ such that (C1) holds and

(C2′) for each $\mathfrak{p}|p$, $\pi_{\mathfrak{p}} = \mathrm{Ind}_B^G \theta_{\mathfrak{p}}$ is spherical, satisfies the hypotheses of Proposition 2.7, and $\tilde{\pi}_{\mathfrak{p}} = (\pi_{\mathfrak{p}}, \alpha_{\mathfrak{p}})$ is the Shalika Q-refinement from that result.

By Proposition 2.7, (C2) is automatic from (C2′). The Q-refined RACARs $\tilde{\pi}$ described in Theorems A, B and C of the introduction satisfy (C1-2′), hence (C1-2).

In general $\alpha_{\mathfrak{p}}$ is not $\mathfrak{p}$-integral. We define weight λ integral normalisations

$$U_{\mathfrak{p}}^{\circ} = \lambda(t_{\mathfrak{p}})U_{\mathfrak{p}}, \quad \alpha_{\mathfrak{p}}^{\circ} = \lambda(t_{\mathfrak{p}})\alpha_{\mathfrak{p}}. \tag{2.19}$$

We justify this in §3.3. A Q-refinement $\tilde{\pi}_{\mathfrak{p}}$ is equivalent to a choice of $U_{\mathfrak{p}}^{\circ}$-eigenvalue $\alpha_{\mathfrak{p}}^{\circ}$ on $\pi_{\mathfrak{p}}^{J_{\mathfrak{p}}}$, and $\pi_{\mathfrak{p}}^{J_{\mathfrak{p}}}[\![U_{\mathfrak{p}} - \alpha_{\mathfrak{p}}]\!] = \pi_{\mathfrak{p}}^{J_{\mathfrak{p}}}[\![U_{\mathfrak{p}}^{\circ} - \alpha_{\mathfrak{p}}^{\circ}]\!]$. Occasionally we abuse notation and write $\tilde{\pi}_{\mathfrak{p}} = (\pi_{\mathfrak{p}}, \alpha_{\mathfrak{p}}^{\circ})$.

2.9. The p-refined Hecke algebra. Let $\tilde{\pi}$ satisfy (C1-2), and let $K \subset G(\mathbf{A}_f)$ be an open compact subgroup with

$$K = \textstyle\prod_v K_v \ \text{ s.t. } K_v = \mathrm{GL}_{2n}(\mathcal{O}_v) \text{ for } v \notin S \cup \{\mathfrak{p}|p\}, K_{\mathfrak{p}} = J_{\mathfrak{p}} \text{ for } \mathfrak{p}|p, \text{ and } \pi_f^K \neq 0. \tag{2.20}$$

Recall $\mathcal{H}'$ and ψ_π (which implicitly are defined at level K) from §2.4.

Definition 2.9. Define $\mathcal{H} = \mathcal{H}'[U_{\mathfrak{p}}^{\circ} : \mathfrak{p}|p]$. Let $E = \mathbf{Q}(\tilde{\pi}, \eta)$ be the number field generated by the Hecke field of π_f, the rationality field of η, and $\alpha_{\mathfrak{p}}^{\circ}$ for $\mathfrak{p}|p$. The character ψ_π extends to

$$\psi_{\tilde{\pi}} : \mathcal{H} \otimes E \longrightarrow E$$

sending $U_{\mathfrak{p}}^{\circ}$ to $\alpha_{\mathfrak{p}}^{\circ}$. Let $\mathfrak{m}_{\tilde{\pi}} := \ker(\psi_{\tilde{\pi}})$. If M is a finite dimensional vector space with an $\mathcal{H}$-action, the localisation $M_{\mathfrak{m}_{\tilde{\pi}}}$ is the generalised eigenspace at $\psi_{\tilde{\pi}}$, i.e.

$$M_{\mathfrak{m}_\pi}[\![U_{\mathfrak{p}}^{\circ} - \alpha_{\mathfrak{p}}^{\circ} : \mathfrak{p}|p]\!] \subset M_{\mathfrak{m}_\pi}.$$

2.10. Automorphic cohomology classes and periods. Recall in Remark 2.4 we fixed an intertwining $\mathcal{S}_{\psi_f}^{\eta_f}$ of π_f. For $\epsilon \in \{\pm 1\}^{\Sigma}$, composing $(\mathcal{S}_{\psi_f}^{\eta_f})^{-1}$ and (2.9) we obtain a $\mathcal{H}$-equivariant isomorphism

$$\Theta^{K,\epsilon} : \mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K) \xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^t(S_K, \mathcal{V}_\lambda^\vee(\mathbf{C}))_{\mathfrak{m}_\pi}^{\epsilon};$$

further composing with (2.10), we obtain a p-adic analogue

$$\Theta_{i_p}^{K,\epsilon} : \mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K) \xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))_{\mathfrak{m}_\pi}^{\epsilon}, \tag{2.21}$$

which is again $\mathcal{H}$-equivariant.

Finally we descend to rational coefficients. Recall the number field E from Definition 2.9. We have a natural action of $\mathrm{Aut}(\mathbf{C})$ on $\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f)$ (see [45, §3.7]), endowing it with an E-structure $\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f, E)$ by [45, Lem. 3.8.1]. We may (and do) take W_f^{FJ} to be an element of $\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f, E)$ (see [45, Lem. 3.9.1]). By [33, Prop. 3.1], [45, Prop. 4.2.1] and [54, §4.4], there exist complex periods Ω_π^ϵ such that $\Theta^{K,\epsilon}/\Omega_\pi^\epsilon$ is $\mathrm{Aut}(\mathbf{C})$-equivariant. In particular, if $L/\mathbf{Q}_p$ is a finite extension containing $i_p(E)$, then

$$\begin{array}{ccccc} \mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K, E) & \xrightarrow{\Theta^{K,\epsilon}/\Omega_\pi^\epsilon} & \mathrm{H}_{\mathrm{c}}^t(S_K, \mathcal{V}_\lambda^\vee(E))_{\mathfrak{m}_\pi}^{\epsilon} & \xhookrightarrow{(2.10)} & \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{V}_\lambda^\vee(L))_{\mathfrak{m}_\pi}^{\epsilon} \\ \big\downarrow & & & & \big\downarrow \\ \mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K) & & \xrightarrow[\sim]{\Theta_{i_p}^{K,\epsilon}/i_p(\Omega_\pi^\epsilon)} & & \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))_{\mathfrak{m}_\pi}^{\epsilon} \end{array} \tag{2.22}$$

commutes, where the vertical arrows are the natural inclusions.

Assume that $\tilde{\pi}$ satisfies Conditions 2.8; we now produce specific cohomology classes attached to $\tilde{\pi}$. At each finite place v of F, in [45, §6.5] the authors define a (sufficiently small) open compact subgroup $K_v \subset \mathrm{GL}_{2n}(F_v)$ such that there exists a Friedberg–Jacquet test vector $W_v^{\mathrm{FJ}} \in$

$\mathcal{S}_{\psi_v}^{\eta_v}(\pi_v)^{K_v}$ as in (2.15). As in [38], we can (and do) take $K_v = \mathrm{GL}_{2n}(\mathcal{O}_v)$ whenever π_v is spherical, and define

$$K(\tilde{\pi}) := \prod_{\mathfrak{p}|p} J_{\mathfrak{p}} \cdot \prod_{v\nmid p} K_v \subset G(\mathbf{A}_f). \tag{2.23}$$

Note $K(\tilde{\pi})$ satisfies (2.20). For $\mathfrak{p}|p$, let $W_{\mathfrak{p}}$ be a generator of the line in (2.18), normalised so that $W_{\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}) = 1$. Write

$$W_f^{\mathrm{FJ}} = \otimes_{\mathfrak{p}|p} W_{\mathfrak{p}} \otimes_{v\nmid p\infty} W_v^{\mathrm{FJ}} \in \mathcal{S}_{\psi_f}^{\eta_f}(\pi_f)^{K(\tilde{\pi})}.$$

Definition 2.10. Let

$$\phi_{\tilde{\pi}}^{\epsilon} = \Theta_{i_p}^{K(\tilde{\pi}),\epsilon}(W_f^{\mathrm{FJ}})/i_p(\Omega_{\pi}^{\epsilon}) \in \mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde{\pi})}, \mathscr{V}_{\lambda}^{\vee}(L))_{\mathfrak{m}_{\pi}}^{\epsilon}.$$

This is precisely the class defined in [38, §4.3.1], where the scaling by Ω_{π}^{ϵ} is implicit. Note that by construction, the class $\phi_{\tilde{\pi}}^{\epsilon}$ is a $U_{\mathfrak{p}}$-eigenclass with eigenvalue $\alpha_{\mathfrak{p}}$ for all $\mathfrak{p}|p$ (see also [38, Lem. 3.6]), thus lies in the p-refined generalised eigenspace $\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{V}_{\lambda}^{\vee}(L))_{\mathfrak{m}_{\tilde{\pi}}}^{\epsilon}$.

3. Overconvergent cohomology and classicality

We recall the Q-parahoric overconvergent cohomology and non-Q-critical slope conditions of [19], while making the theory explicit in our setting.

3.1. Weight spaces.

Recall $X^*(T)$, $X_0^*(T)$, $X^*(H)$ and $X_0^*(H)$ from §2.2.

Definition 3.1 (Weights for T)**.** The *weight space* $\mathscr{W}^G$ for G is the rigid analytic space whose L-points, for $L \subset \mathbf{C}_p$ any sufficiently large extension of $\mathbf{Q}_p$, are given by

$$\mathscr{W}^G(L) = \mathrm{Hom}_{\mathrm{cont}}(T(\mathbf{Z}_p), L^\times).$$

This space contains the set $X_+^*(T)$ of dominant integral weights in a natural way. We call any element of this subspace an *algebraic weight*.

A weight $\lambda \in \mathscr{W}^G$ decomposes as $\lambda = (\lambda_1, ..., \lambda_{2n})$, where each λ_i is a character of $(\mathcal{O}_{F,p})^\times$. We see that $\mathscr{W}^G$ has dimension $2dn$.

Definition 3.2. Let $\mathscr{W}_0^G$ be the $(dn+1)$-dimensional *pure weight space*, that is the Zariski closure of the pure, dominant, integral weights $X_0^*(T)$ in $\mathscr{W}^G$. We have

$$\mathscr{W}_0^G(L) := \{\lambda \in \mathscr{W}^G(L) \mid \exists \mathsf{w}_\lambda \in \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, L^\times) \text{ s.t. } \lambda_i \cdot \lambda_{2n+1-i} = \mathsf{w}_\lambda \circ \mathrm{N}_{F/\mathbf{Q}} \ \forall\ 1 \leqslant i \leqslant n\}.$$

In §1.2.3, we highlighted the flexibility of using parahoric distributions: much weaker notions of finite slope families and non-criticality. This comes at the cost of less flexibility in variation, as such distributions vary only over the following $(d+1)$-dimensional subspaces of $\mathscr{W}_0^G$.

Definition 3.3 (Weights for Q)**.** Define $\mathscr{W}^Q \subset \mathscr{W}^G$ to be the rigid subspace whose L-points are continuous characters that factor through a character $H(\mathbf{Z}_p) \to L^\times$. Let $\mathscr{W}_0^Q := \mathscr{W}^Q \cap \mathscr{W}_0^G$ be the pure subspace. These are the Zariski closures of $X^*(H)$ and $X_0^*(H)$ in $\mathscr{W}^G$.

The space $\mathscr{W}^Q$ is the subspace of $\mathscr{W}^G$ where

$$\lambda_1 = \cdots = \lambda_n (= \nu_1, \text{ say}) \qquad \text{and} \qquad \lambda_{n+1} = \cdots = \lambda_{2n} (= \nu_2, \text{ say}).$$

The association $\lambda \mapsto (\nu_1, \nu_2)$ identifies $\mathscr{W}^Q$ isomorphically with the $2d$-dimensional (Hilbert) weight space of $\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_2$; the $(d+1)$-dimensional pure subspace $\mathscr{W}_0^Q$ is canonically identified with the pure Hilbert weights.

Definition 3.4. For $\lambda_\pi \in X_0^*(T)$ a pure, dominant, algebraic 'base' weight (implicitly, the weight of an automorphic representation π) let

$$\mathscr{W}_{\lambda_\pi}^Q := \{\lambda \in \mathscr{W}_0^G \mid \lambda\lambda_\pi^{-1} \in \mathscr{W}_0^Q\} = \lambda_\pi \mathscr{W}_0^Q \subset \mathscr{W}_0^G.$$

Remark 3.5. To get non-trivial weight λ local systems on S_K we need $\lambda(Z(\mathbf{Q}) \cap K) = 1$. If π is a RACAR of weight λ_π, and K satisfies (2.20) for π, this condition is satisfied by existence of an automorphic form fixed by K. It is hence also true for all λ in a sufficiently small affinoid neighbourhood $\Omega \subset \mathscr{W}_{\lambda_\pi}^Q$ of λ_π, as such λ are pure and hence

$$\lambda(Z(\mathbf{Q}) \cap K) \subset (\mathsf{w} \circ \mathrm{N}_{F/\mathbf{Q}})^n(\mathcal{O}_F^\times) \subset \{\pm 1\}$$

is discrete. As all of our arguments are local in $\mathscr{W}_{\lambda_\pi}^Q$, for the rest of the paper we will always assume this condition is satisfied for the levels K and over the affinoids Ω we work with.

3.2. Parahoric distribution modules. Recall for $L/\mathbf{Q}_p$ sufficiently large, $V_\lambda(L)$ is the algebraic induction $\mathrm{Ind}_{B(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)}\lambda$. Typically overconvergent cohomology coefficients are dual to the locally analytic induction $\mathcal{A}_\lambda^B$ of λ to the Iwahori subgroup. We define Q-parahoric analogues.

If $X \subset \mathbf{Q}_p^r$ is compact and R is a $\mathbf{Q}_p$-Banach algebra, let $\mathcal{A}(X,R)$ be the space of locally analytic functions $X \to R$, and

$$\mathcal{D}(X,R) := \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}(X,R),R)$$

be its topological R-dual. If W is a finite Banach R-module, then we say a function $f : X \to W$ is locally analytic if it is an element of $\mathcal{A}(X,R) \otimes_R W$, and write $\mathcal{A}(X,W)$ for the space of such functions. (These definitions are explained in detail in [19, §3.2.2]).

3.2.1. Parahoric algebraic induction modules. As motivation, we first give a parahoric description of V_λ. Let $G_n = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\mathrm{GL}_n$ and recall $H = G_n \times G_n$. Considering $\lambda \in X_0^*(T)$ as a weight for H, the algebraic representation of H of highest weight λ is

$$V_\lambda^H(L) = \mathrm{Ind}_{B^-(\mathbf{Z}_p)\cap H(\mathbf{Z}_p)}^{H(\mathbf{Z}_p)}\lambda = V_{\lambda'}^{G_n}(L) \otimes V_{\lambda''}^{G_n}(L),$$

where $\lambda' = (\lambda_1, ..., \lambda_n)$ and $\lambda'' = (\lambda_{n+1}, ..., \lambda_{2n})$. Again, $V_\lambda^H(L)$ is the space of algebraic $f_H : H(\mathbf{Z}_p) \to L$ satisfying the H-analogue of (2.1).

The action of $H(\mathbf{Z}_p)$ on $V_\lambda^H(L)$ yields a homomorphism

$$\langle\cdot\rangle_\lambda : H(\mathbf{Z}_p) \to \mathrm{Aut}(V_\lambda^H(L)). \tag{3.1}$$

We say a function $\mathcal{F} : G(\mathbf{Z}_p) \to V_\lambda^H(L)$ is *algebraic* if it is an element of $L[G] \otimes_L V_\lambda^H(L)$. Let

$$\mathrm{Ind}_{Q^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} V_\lambda^H(L) := \{\mathcal{F} : G(\mathbf{Z}_p) \to V_\lambda^H(L) \mid \mathcal{F} \text{ algebraic}, \mathcal{F}(n_Q^- hg) = \langle h\rangle_\lambda \mathcal{F}(g) \quad \forall n_Q^- \in N_Q^-(\mathbf{Z}_p), h \in H(\mathbf{Z}_p), g \in G(\mathbf{Z}_p)\}. \tag{3.2}$$

This has a $G(\mathbf{Z}_p)$ action by $(\gamma \cdot \mathcal{F})(g) = \mathcal{F}(g\gamma)$.

The following lemma says 'algebraic induction is transitive'.

Lemma 3.6. *[53, §I.3.5]. There is a canonical isomorphism of $G(\mathbf{Z}_p)$-representations*

$$\mathrm{Ind}_{Q^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} V_\lambda^H(L) \xrightarrow{\sim} V_\lambda(L), \qquad \mathcal{F} \mapsto [g \mapsto \mathcal{F}(g)(\mathrm{id}_H)].$$

Proof. Evaluation at the identity $\mathrm{id}_H \in H(\mathbf{Z}_p)$ induces a linear map $V_\lambda^H(L) \to L$, and this induces a map

$$i : L[G] \otimes_L V_\lambda^H(L) \to L[G],$$

which we interpret as a map

$$\begin{aligned} i : \big\{\text{algebraic } G(\mathbf{Z}_p) \to V_\lambda^H(L)\big\} &\longrightarrow \big\{\text{algebraic } G(\mathbf{Z}_p) \to L\big\}, \\ \mathcal{F} &\longmapsto [g \mapsto (\mathcal{F}(g))(\mathrm{id}_H)]. \end{aligned}$$

If $\mathcal{F} \in \mathrm{Ind}_{Q^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} V_\lambda^H(L)$, then a straightforward computation (as in [19, Prop. 4.8]) shows $i(\mathcal{F})$ satisfies (2.1). Combining, we see i restricts to a well-defined map

$$i : \mathrm{Ind}_{Q^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} V_\lambda^H(L) \longrightarrow V_\lambda(L).$$

If $i(\mathcal{F}) = 0$, then for any $g \in G(\mathbf{Z}_p), h \in H(\mathbf{Z}_p)$, we have

$$\begin{aligned} \big[\mathcal{F}(g)\big](h) &= \big[\langle h\rangle_\lambda \mathcal{F}(g)\big](\mathrm{id}_H) \\ &= \big[\mathcal{F}(hg)\big](\mathrm{id}_H) = \big[i(\mathcal{F})\big](hg) = 0, \end{aligned}$$

so $\mathcal{F} = 0$ and i is injective. The map i is evidently $G(\mathbf{Z}_p)$-equivariant, so i identifies $\mathrm{Ind}_{Q^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} V_\lambda^H(L)$ with a $G(\mathbf{Z}_p)$-subrepresentation of $V_\lambda(L)$; but the latter is irreducible, so i is an isomorphism. □

3.2.2. Parahoric analytic induction modules. We have

$$V_\lambda \cong \mathrm{Ind}_{Q^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} \mathrm{Ind}_{B^- \cap H(\mathbf{Z}_p)}^{H(\mathbf{Z}_p)} \lambda$$

by Lemma 3.6. To define Q-parahoric analogues of $\mathcal{A}_\lambda^B$, in Lemma 3.6 we replace the algebraic induction from $Q^-(\mathbf{Z}_p)$ with locally analytic induction. Let $J_p := \prod_{\mathfrak{p}|p} J_\mathfrak{p}$ denote the parahoric subgroup for Q, as defined in (2.16). Let $\mathcal{A}_\lambda^Q(L)$ denote the space of functions

$$\big[f : J_p \to V_\lambda^H(L)\big] \in \mathcal{A}(J_p, V_\lambda^H(L))$$

such that

$$f(n^- hg) = \langle h\rangle_\lambda f(g) \text{ for all } n^- \in N_Q^-(\mathbf{Z}_p) \cap J_p, h \in H(\mathbf{Z}_p), \text{ and } g \in J_p.$$

Again restriction identifies $\mathcal{A}_\lambda^Q(L)$ with $\mathcal{A}(N_Q(\mathbf{Z}_p), V_\lambda^H(L))$. Let

$$\mathcal{D}_\lambda^Q(L) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}_\lambda^Q(L), L)$$

be the topological dual; this is a compact Fréchet space [19, §3.2.3].

Remark 3.7. Note that any $n \in N(\mathbf{Z}_p)$ can be uniquely written as a product

$$n = h \cdot n_Q = \left(\begin{array}{cc|cc} 1 & x_{ij} & & \\ & \ddots & & \\ \hline & & \ddots & x_{k\ell} \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & & y_{1,n+1} & \cdots \\ & \ddots & \cdots & y_{n,2n} \\ \hline & & \ddots & \\ & & & 1 \end{array}\right),$$

of $h \in H(\mathbf{Z}_p)$ and $n_Q \in N_Q(\mathbf{Z}_p)$, where $x_{ij}, y_{ij} \in \mathcal{O}_{F,p}$. Then for any $f \in \mathcal{A}_\lambda^B$ the restriction $f|_{N(\mathbf{Z}_p)}$ is locally analytic in the $x_{ij,\sigma}$ and $y_{ij,\sigma}$ (thought of as $\mathbf{Z}_p$-coordinates on $N(\mathbf{Z}_p)$) . On the other hand, for any $f \in \mathcal{A}_\lambda^Q$ the restriction $f|_{N_Q(\mathbf{Z}_p)}$ is a locally analytic function in the $y_{ij,\sigma}$ with coefficients in V_λ^H. As V_λ^H can be realised as a space of polynomials in the $x_{ij,\sigma}$, one sees that $\mathcal{A}_\lambda^Q$ is an intermediate space between V_λ and $\mathcal{A}_\lambda^B$. A precise description of the natural inclusion $\mathcal{A}_\lambda^Q \subset \mathcal{A}_\lambda^B$ is given in [19, Props. 4.9, 4.11].

Notation 3.8. Since throughout we will only be interested in Q-parahoric distributions, we will henceforth suppress superscript Q's and write $\mathcal{A}_\lambda := \mathcal{A}_\lambda^Q$ and $\mathcal{D}_\lambda := \mathcal{D}_\lambda^Q$.

3.2.3. Distributions in families. Let $\Omega \subset \mathscr{W}^Q_{\lambda_\pi}$ be an affinoid, for a fixed $\lambda_\pi \in X_0^*(T)$. If $\lambda \in \Omega$ is algebraic, then by definition $\lambda\lambda_\pi^{-1} \in \mathscr{W}_0^Q$ and there is an isomorphism

$$V_\lambda^H = V_{\lambda_\pi}^H \otimes \lambda\lambda_\pi^{-1} \tag{3.3}$$

of $H(\mathbf{Z}_p)$-modules [19, Lem. 3.8]. In particular, the underlying spaces of V_λ^H and $V_{\lambda_\pi}^H$ are the same, allowing analytic variation of the representation V_λ^H as λ varies in an affinoid of $\mathscr{W}^Q_{\lambda_\pi}$. This is crucial for variation and is not true of any higher-dimensional affinoid neighbourhoods of λ. If we use B rather than Q, the analogue of V_λ^H is the 1-dimensional character λ, which can evidently vary in an affinoid subspace of the entire weight space $\mathscr{W}^G$.

The space $\Omega_0 := \{\lambda\lambda_\pi^{-1} : \lambda \in \Omega\}$ is an affinoid in $\mathscr{W}_0^Q \subset \mathscr{W}^G$.

Lemma 3.9. *The character $\chi_{\Omega_0} : H(\mathbf{Z}_p) \longrightarrow \mathcal{O}_{\Omega_0}^\times$ given by $h \mapsto [\lambda_0 \mapsto \lambda_0(h)]$ is locally analytic.*

Proof. This is proved in [19, §3.2.6] using [28, Prop. 8.3]. □

As χ_{Ω_0} is a character of $H(\mathbf{Z}_p)$, it factors through its abelianisation

$$H(\mathbf{Z}_p) \xrightarrow{\det} (\mathcal{O}_{F,p}^\times)^2,$$

so there exists a character $(\chi^1_{\Omega_0}, \chi^2_{\Omega_0})$ of $(\mathcal{O}_{F,p}^\times)^2$ such that

$$\chi_{\Omega_0}(h_1, h_2) = (\chi^1_{\Omega_0} \circ \det(h_1)) \cdot (\chi^2_{\Omega_0} \circ \det(h_2)).$$

As Ω_0 is a subspace of the pure weights, there exists

$$\mathsf{w}_{\Omega_0} : \mathbf{Z}_p^\times \to \mathcal{O}_{\Omega_0}^\times$$

such that

$$\chi^1_{\Omega_0}(x) \cdot \chi^2_{\Omega_0}(x) = \mathsf{w}_{\Omega_0} \circ \mathrm{N}_{F/\mathbf{Q}}(x)$$

for all $x \in \mathcal{O}_{F,p}^\times$, and hence

$$\chi_{\Omega_0}(h, h) = \mathsf{w}_{\Omega_0} \circ \mathrm{N}_{F/\mathbf{Q}}(\det(h)). \tag{3.4}$$

If $\lambda_0 \in \Omega_0$, then evaluation at λ_0 sends w_{Ω_0} to w_{λ_0}, as defined in Definition 3.2, so w_{Ω_0} interpolates purity weights over Ω_0.

Now define $V_\Omega^H := V_{\lambda_\pi}^H(L) \otimes_L \mathcal{O}_{\Omega_0}$, a free $\mathcal{O}_{\Omega_0}$-module of finite rank, and a homomorphism

$$\begin{aligned} \langle\cdot\rangle_\Omega : H(\mathbf{Z}_p) &\longrightarrow \mathrm{Aut}\big(V_{\lambda_\pi}^H(L)\big) \otimes \mathcal{O}_{\Omega_0}^\times \subset \mathrm{Aut}\big(V_\Omega^H\big), \\ h &\longmapsto \langle h\rangle_{\lambda_\pi} \otimes \chi_{\Omega_0}(h). \end{aligned} \tag{3.5}$$

This makes V_Ω^H into an $H(\mathbf{Z}_p)$-representation.

Definition 3.10. Let $\lambda \in \Omega(L)$, and let $\lambda_0 = \lambda\lambda_\pi^{-1} \in \Omega_0(L)$. Define a map $\mathrm{sp}_{\lambda_0} : \mathcal{O}_{\Omega_0} \to L$ by evaluating functions at λ_0. This induces a map

$$\mathrm{sp}_\lambda : V_\Omega^H \xrightarrow{\mathrm{id}\otimes \mathrm{sp}_{\lambda_0}} V_{\lambda_\pi}^H(L) \otimes \lambda\lambda_\pi^{-1} \xrightarrow[\sim]{(3.3)} V_\lambda^H(L). \tag{3.6}$$

Since $\mathrm{sp}_{\lambda_0} \circ \chi_{\Omega_0} = \lambda_0$ by (3.3), this map is $H(\mathbf{Z}_p)$-equivariant. In particular, V_Ω^H interpolates the representations V_λ^H as λ varies in Ω (where if λ is non-algebraic, $V_\lambda^H := V_{\{\lambda\}}^H$).

Choosing λ_π fixes an isomorphism $\Omega \xrightarrow{\sim} \Omega_0$, $\lambda \mapsto \lambda_\pi^{-1}\lambda$. This induces $\mathcal{O}_\Omega \xrightarrow{\sim} \mathcal{O}_{\Omega_0}$, compatible with specialisation maps. Under this we may define characters

$$\begin{aligned} \chi_\Omega &:= \lambda_\pi \cdot \chi_{\Omega_0} : H(\mathbf{Z}_p) \to \mathcal{O}_\Omega^\times, \\ \mathsf{w}_\Omega &:= \mathsf{w}_{\lambda_\pi} \cdot \mathsf{w}_{\Omega_0} : \mathbf{Z}_p^\times \to \mathcal{O}_\Omega^\times \end{aligned} \tag{3.7}$$

such that evaluation at $\lambda \in \Omega$ sends χ_Ω to λ and w_Ω to w_λ. Henceforth we work only with Ω, suppressing Ω_0, and implicitly any transfer of structure is with respect to this identification.

Definition 3.11. Define $\mathcal{A}_\Omega$ to be the space of functions

$$\big[f : J_p \to V_\Omega^H\big] \in \mathcal{A}(J_p, V_\Omega^H)$$

such that

$$f(n^- hg) = \langle h\rangle_\Omega f(g) \text{ for all } n^- \in N_Q^-(\mathbf{Z}_p) \cap J_p, h \in H(\mathbf{Z}_p), \text{ and } g \in J_p. \tag{3.8}$$

Define $\mathcal{D}_\Omega := \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}_\Omega, \mathcal{O}_\Omega)$. This is a compact Fréchet $\mathcal{O}_\Omega$-module (see [19, Lem. 3.16]).

Remark 3.12. As in [19, Rem. 3.18], if $\Omega' \subset \Omega$ is a closed affinoid, then

$$\mathcal{D}_\Omega \otimes_{\mathcal{O}_\Omega} \mathcal{O}_{\Omega'} \cong \mathcal{D}_{\Omega'}.$$

As a special case, suppose $\Omega' = \{\lambda\}$ is a single point, whence $\mathcal{O}_{\Omega'} = L$ is a field, of the form $\mathcal{O}_\Omega/\mathfrak{m}_\lambda$ for $\mathfrak{m}_\lambda \subset \mathcal{O}_\Omega$ the maximal ideal attached to λ. The map $\mathrm{sp}_\lambda : \mathcal{O}_\Omega \to L$ is reduction modulo $\mathfrak{m}_\lambda$. Then we see

$$\mathrm{sp}_\lambda(\mathcal{D}_\Omega) := \mathcal{D}_\Omega \otimes_{\mathcal{O}_\Omega} \mathcal{O}_\Omega/\mathfrak{m}_\lambda \cong \mathcal{D}_\lambda(L).$$

In particular, $\mathcal{D}_\Omega$ interpolates $\mathcal{D}_\lambda$ as λ varies in Ω.

3.3. The action of U_p° and slope decompositions.

Fix any open compact subgroup $K \subset G(\mathbf{A}_f)$ such that $K_p \subset J_p$ (e.g. $K = K(\tilde{\pi})$). Let Ω be an affinoid in $\mathscr{W}_{\lambda_\pi}^Q$; we allow $\Omega = \{\lambda\}$ a single weight, in which case $\mathcal{O}_\Omega = L$. We have a natural left action of J_p on $\mathcal{A}_\Omega$ by

$$(k * f)(g) = f(gk), \qquad k \in J_p, f \in \mathcal{A}_\Omega, g \in J_p,$$

inducing a dual left action of J_p on $\mathcal{D}_\Omega$ by $(k * \mu)(f) := \mu(k^{-1} * f)$. Thus $\mathcal{D}_\Omega$ is a left K-module (via projection to K_p), giving a local system $\mathscr{D}_\Omega$ on the space S_K as in §2.3.2. (For this to be non-trivial, we need the centre of $G(\mathbf{Q}) \cap KK_\infty^\circ$ to act trivially. This holds by definition of the action of K on $\mathcal{D}_\Omega$, and since for Ω sufficiently small, one has $\chi_\Omega(Z(\mathbf{Q}) \cap K) = 1$ by Remark 3.5).

Recall $t_{\mathfrak{p}} = \iota(\varpi_{\mathfrak{p}} I_n, I_n)$. Note

$$t_{\mathfrak{p}} N_Q(\mathbf{Z}_p) t_{\mathfrak{p}}^{-1} \subset N_Q(\mathbf{Z}_p).$$

For any $f \in \mathcal{A}_\Omega$, define a function

$$t_{\mathfrak{p}}^{-1} * f : N_Q(\mathbf{Z}_p) \to V_\Omega^H$$

sending $n \in N_Q(\mathbf{Z}_p)$ to $f(t_{\mathfrak{p}} n t_{\mathfrak{p}}^{-1})$. As $H(\mathbf{Z}_p)$ commutes with $t_{\mathfrak{p}}$, using (3.8) and parahoric decomposition, $t_{\mathfrak{p}}^{-1} * f$ extends to a unique function in $\mathcal{A}_\Omega$.

Let $\Delta_p \subset G(\mathbf{Q}_p)$ be the semigroup generated by J_p and $t_{\mathfrak{p}}$ for $\mathfrak{p}|p$. One checks (e.g. as in [81, §3.1.3]) that the actions of J_p and $t_{\mathfrak{p}}^{-1}$ on $\mathcal{A}_\Omega$ extend to a left action of Δ_p^{-1} on $\mathcal{A}_\Omega$. We get a dual left action of Δ_p on $\mathcal{D}_\Omega$ by $(\delta * \mu)(f) = \mu(\delta^{-1} * f)$. We then equip the cohomology groups $\mathrm{H}_c^i(S_K, \mathscr{D}_\Omega)$ with an $\mathcal{O}_\Omega$-linear action of the Hecke operators $U_{\mathfrak{p}}^\circ := [K_{\mathfrak{p}} t_{\mathfrak{p}} K_{\mathfrak{p}}]$ in the usual way.

Remark 3.13. To justify the notation, note the $*$-action on $\mathcal{D}_\lambda(L)$ preserves its natural integral subspace (see [19, §3.2.5]). Also, the $*$-action on $\mathcal{A}_\lambda$ preserves the subspace V_λ, so dualising we see $\mathcal{D}_\lambda(L)$ admits $V_\lambda^\vee(L)$ as a $*$-stable quotient. We thus obtain an induced $*$-action on $V_\lambda^\vee(L)$.

On $V_\lambda^\vee(L)$, we also have the natural algebraic $\cdot$-action of $G(\mathbf{Q}_p)$ from §2.3.2. The $*$- and $\cdot$-actions of Δ_p on $V_\lambda^\vee(L)$ coincide for $J_p \subset \Delta_p$, so give the same p-adic local system $\mathscr{V}_\lambda^\vee(L)$. However, the actions of $t_{\mathfrak{p}}$ are different; analogously to [19, Rem. 3.23] one computes that

$$t_{\mathfrak{p}} * \mu = \lambda(t_{\mathfrak{p}}) \cdot \big(t_{\mathfrak{p}} \cdot \mu\big), \quad \mu \in V_\lambda^\vee(L). \tag{3.9}$$

The two actions induce Hecke operators $U_{\mathfrak{p}}^*$ and $U_{\mathfrak{p}}^{\cdot}$ on the classical cohomology, related by

$$U_{\mathfrak{p}}^* = \lambda(t_{\mathfrak{p}})U_{\mathfrak{p}}^{\cdot} = \varpi_{\mathfrak{p}}^{\sum_{\sigma\in\Sigma(\mathfrak{p})}\sum_{i=1}^n \lambda_{\sigma,i}} U_{\mathfrak{p}}^{\cdot}.$$

The isomorphism (2.21) is equivariant for the natural $U_{\mathfrak{p}}$-operator on π_f^K and the operator $U_{\mathfrak{p}}^{\cdot}$ on cohomology, so the class $\phi_{\tilde\pi}^{\epsilon}$ from Definition 2.10 is an eigenclass with $U_{\mathfrak{p}}^{\cdot}$-eigenvalue $\alpha_{\mathfrak{p}}$. However $U_{\mathfrak{p}}^{\cdot}$ does not preserve integrality, whilst $U_{\mathfrak{p}}^*$ does. Because of this it is standard to write

$$U_{\mathfrak{p}} \coloneqq U_{\mathfrak{p}}^{\cdot} = \text{ the usual automorphic Hecke operator on cohomology,}$$
$$U_{\mathfrak{p}}^{\circ} \coloneqq U_{\mathfrak{p}}^{*} = \lambda(t_{\mathfrak{p}})U_{\mathfrak{p}} \text{ its ‘optimal integral normalisation’.}$$

Thus $\phi_{\tilde\pi}^{\epsilon}$ is a $U_{\mathfrak{p}}^{\circ}$-eigenclass with eigenvalue $\alpha_{\mathfrak{p}}^{\circ} \coloneqq \lambda(t_{\mathfrak{p}})\alpha_{\mathfrak{p}}$. As $U_{\mathfrak{p}}^{\circ}$ preserves integrality, $v_p(\alpha_{\mathfrak{p}}^{\circ}) \geqslant 0$.

The $U_{\mathfrak{p}}$ and $U_{\mathfrak{p}}^{\circ}$ operators of [38] coincide with ours. Their $\bullet$-action on $V_\lambda^\vee$ is our $*$-action.

Let $t_p = \iota(pI_n, I_n)$. As

$$\bigcap_{i\geqslant 0} t_p^i N_Q(\mathbf{Z}_p) t_p^{-i} = 1,$$

we have $t_p \in T_Q^{++}$ in the notation* of [19, §2.5], and (via the $*$-action) we get a Q-controlling operator $U_p^{\circ} \coloneqq [K_p t_p K_p]$ on the cohomology. By §3.5 *ibid.*, for any $h \in \mathbf{Q}_{\geqslant 0}$, up to shrinking Ω the $\mathcal{O}_\Omega$-module $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)$ admits a slope $\leqslant h$ decomposition with respect to U_p° (see [46, Def. 2.3.1]). We let $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)^{\leqslant h}$ denote the subspace of elements of slope at most h, and note that it is an $\mathcal{O}_\Omega$-module of finite type.

3.4. Non-critical slope conditions for Q. Let $\lambda \in X_0^*(T)$ be a pure dominant integral weight and K as in §3.3. The natural inclusion of $V_\lambda(L) \subset \mathcal{A}_\lambda(L)$ induces dually a surjection $\mathcal{D}_\lambda(L) \longrightarrow V_\lambda^\vee(L)$, which is equivariant for the $*$-actions of Δ_p. This induces a map

$$r_\lambda : \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(L)) \longrightarrow \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L)), \tag{3.10}$$

equivariant for the $*$-actions of Δ_p on both sides; hence by Remark 3.13, it is equivariant for the actions of $U_{\mathfrak{p}}^{\circ}$ on both sides.

Let $\tilde\pi$ be a Q-refined RACAR of $G(\mathbf{A})$ of weight λ and $h \gg 0$. As $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(L))^{\leqslant h}$ is a finite dimensional vector space, the localisation $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(L))^{\leqslant h}_{\mathfrak{m}_{\tilde\pi}}$ is the generalised eigenspace in $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(L))$ where the Hecke operators act with the same eigenvalues as on $\tilde\pi$ (see §2.9). Abusing notation, we drop the $\leqslant h$ and just write $\mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(L))_{\mathfrak{m}_{\tilde\pi}}$ for this generalised eigenspace.

Definition 3.14. Let $\tilde\pi$ be a Q-refined RACAR of $G(\mathbf{A})$ of weight λ. We say $\tilde\pi$ is *non-Q-critical (at level K)* if the restriction of r_λ to the generalised eigenspaces

$$r_\lambda : \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{D}_\lambda(L))_{\mathfrak{m}_{\tilde\pi}} \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L))_{\mathfrak{m}_{\tilde\pi}}$$

is an isomorphism. If K is clear from the context we will not specify it.

We say $\tilde\pi$ is *strongly non-Q-critical* if this is true for all K satisfying (2.20), and also with $\mathrm{H}^\bullet_{\mathrm{c}}$ replaced with $\mathrm{H}^\bullet$ (i.e., if $\tilde\pi$ is non-Q-critical for $\mathrm{H}^\bullet$ *and* for $\mathrm{H}^\bullet_{\mathrm{c}}$ as in [19, Rem. 4.6]).

Recall $\Sigma = \coprod_{\mathfrak{p}|p} \Sigma(\mathfrak{p})$ from §2.1. For $\mathfrak{p}|p$, let $e_{\mathfrak{p}}$ be the ramification degree of $\mathfrak{p}|p$.

Definition 3.15. For $\mathfrak{p}|p$, we say that $\tilde\pi_{\mathfrak{p}} = (\pi_{\mathfrak{p}}, \alpha_{\mathfrak{p}})$ has *non-Q-critical slope* if

$$e_{\mathfrak{p}} \cdot v_p\big(\alpha_{\mathfrak{p}}^{\circ}\big) < \min_{\sigma\in\Sigma(\mathfrak{p})}(1 + \lambda_{\sigma,n} - \lambda_{\sigma,n+1}),$$

where $\alpha_{\mathfrak{p}}^{\circ} = \lambda(t_{\mathfrak{p}})\alpha_{\mathfrak{p}}$. If this holds at all $\mathfrak{p}|p$, we say that $\tilde\pi$ has *non-Q-critical slope.*

*As distributions in [19] are right-modules, all conventions are opposite to those here: see [19, Rem. 4.20].

Theorem 3.16 (Classicality)**.** *If $\tilde{\pi}$ has non-Q-critical slope, then it is strongly non-Q-critical.*

Proof. This is a special case of [19, Thm. 4.4, Rem. 4.6], explained in Examples 4.5 *ibid.*. The root system of $G_{/\mathbf{Q}_p}$ is a disjoint union, indexed by Σ, of copies of the standard root system A_{2n-1} of GL_{2n}, and we denote by $\{\beta_{1,\sigma}, ..., \beta_{2n-1,\sigma}\}$ the simple roots in the copy of A_{2n-1} attached to $\sigma \in \Sigma$. The parabolic Q corresponds to the subset

$$\Delta_Q = \Delta \backslash \{\beta_{n,\sigma} : \sigma \in \Sigma\}$$

of the set Δ of simple roots for $G_{/\mathbf{Q}_p}$. In the notation *ibid.*, if $\alpha_i = \beta_{n,\sigma}$ with $\sigma \in \Sigma(\mathfrak{p})$ for a prime $\mathfrak{p}|p$, we take $t_i = t_{\mathfrak{p}}$. Thus

$$U_i = U_{\mathfrak{p}}^{\circ}, \quad v_p(\alpha_i(t_i)) = 1/e_{\mathfrak{p}},$$

$$\text{and} \quad h^{\mathrm{crit}}(t_i, \alpha_i, \lambda) = (1 + \lambda_{\sigma,n} - \lambda_{\sigma,n+1})/e_{\mathfrak{p}}.$$

Thus Definition 3.15 gives precisely the non-critical slope condition of [19, Thm. 4.4]. Note the level *ibid.* is arbitrary. Any differences in conventions are explained in [19, §4.6] (though the reader is warned that the comparison of $U_{\mathfrak{p}}^{\circ}$ and $U_{\mathfrak{p}}$ was computed incorrectly there; we are using the corrected version in this paper). ☐

Remark 3.17. It is natural to ask if there exist $\tilde{\pi}$ that are non-Q-critical (for some K) but not strongly non-Q-critical. In the case of GL_2, it is conjectured that a modular form is critical if and only if its local Galois representation at p is split. If a Galois-theoretic criterion for non-criticality exists more generally, then one would expect that non-Q-criticality should not depend on the type or level of cohomology, and hence that non-Q-critical implies strongly non-Q-critical.

4. Abstract evaluation maps

We now describe an abstract theory of evaluation maps, which are linear functionals on compactly supported cohomology groups. Recall that $H = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}(\mathrm{GL}_n \times \mathrm{GL}_n)$, with diagonal embedding $\iota : H \hookrightarrow G$. The underlying idea behind evaluation maps is to integrate compactly supported cohomology classes for G over (coverings of) locally symmetric spaces for H, the so-called *automorphic cycles*. The arithmetic importance of such a construction is that when we start with a class attached to a RASCAR π (as in Definition 2.10), the resulting integrals take the shape of relative global zeta integrals for $H \subset G$ that we can relate to L-values of π.

The constructions of [45, 38] treat evaluation maps for cohomology with coefficients in the algebraic local systems attached to $V_\lambda^\vee$. In this chapter, we give a new construction that allows coefficients in much more general local systems.

We describe the automorphic cycles, and their basic properties, in §4.1. Crucially, they have real dimension equal to t, the top degree of cohomology to which RACARs for G contribute. This 'magical numerology' allows one to integrate degree t compactly supported cohomology classes against automorphic cycles, and we make this integration theory precise, with general coefficients, in §4.2. In §4.3, we finally show that our constructions depend functorially on the coefficient system, and track their dependence on the level of the automorphic cycles.

In §5 we will use this abstract theory with classical cohomology groups – those with coefficients in $\mathscr{V}_\lambda^\vee$ – to recover the evaluations of [38], and their connection to Deligne-critical L-values of RASCARs. In §6 we use it to define distribution-valued evaluations on the overconvergent cohomology groups $\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega)$, and hence to p-adically interpolate the evaluations of [38].

4.1. Automorphic cycles.

Definition 4.1. Let $L_H \subset H(\mathbf{A}_f)$ be an open compact subgroup. Define the *automorphic cycle* of level L_H to be the space

$$X_{L_H} := H(\mathbf{Q})\backslash H(\mathbf{A})/L_H L_\infty^\circ,$$

where $L_\infty = H_\infty \cap K_\infty$ for $H_\infty := H(\mathbf{R})$ (note all intersections are taken with respect to ι). Note $Z_G(\mathbf{R}) \cap H_\infty \subsetneq Z_H(\mathbf{R})$, so this is not the locally symmetric space for H. This is denoted $\tilde{S}^H_{L_H}$ in [38], and is a real orbifold of dimension t [38, (23)].

We choose a specific L_H, as in [38, §2.1]. Let $K \subset G(\mathbf{A}_f)$ be an open compact subgroup.

Definition 4.2. (i) Define a matrix $\xi \in \mathrm{GL}_{2n}(\mathbf{A}_F)$ by setting $\xi_v = 1$ for all $v \nmid p$ and

$$\xi_{\mathfrak{p}} = \left(\begin{smallmatrix} I_n & w_n \\ 0 & w_n \end{smallmatrix}\right) \in \mathrm{GL}_{2n}(\mathcal{O}_{F,\mathfrak{p}})$$

for $\mathfrak{p}|p$, where w_n is the antidiagonal $n \times n$ matrix whose (i,j)-th entry is $\delta_{i,n-j+1}$.

(ii) For a multi-exponent $\beta = (\beta_{\mathfrak{p}})_{\mathfrak{p}|p}$, with $\beta_{\mathfrak{p}} \in \mathbf{Z}_{\geqslant 0}$, we write

$$p^\beta = \prod \varpi_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} \qquad \text{and} \qquad t_p^\beta = \prod t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}.$$

Fix an ideal $\mathfrak{m} \subset \mathcal{O}_F$ prime to p. Then we define

$$L_\beta = L^{(p)} \prod_{\mathfrak{p}|p} L_{\mathfrak{p}}^{\beta_{\mathfrak{p}}},$$

where

(L1) away from p,
$$L^{(p)} = \{h \in H(\widehat{\mathbf{Z}}^{(p)}) : h \equiv 1 \,(\mathrm{mod}\, \mathfrak{m})\}$$
is the principal congruence subgroup of level $\mathfrak{m}$,

(L2) at $\mathfrak{p}|p$,
$$L_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} := H(\mathbf{Z}_p) \cap K_{\mathfrak{p}} \cap \xi_{\mathfrak{p}} t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} K_{\mathfrak{p}} t_{\mathfrak{p}}^{-\beta_{\mathfrak{p}}} \xi_{\mathfrak{p}}^{-1}.$$

Let $X_\beta := X_{L_\beta}$ be the *automorphic cycle of level* p^β.

The ideal $\mathfrak{m}$ will always be fixed large enough so that

$$L^{(p)} \text{ is contained in } K^{(p)} \cap H(\mathbf{A}_f), \quad \text{and} \tag{4.1}$$

$$H(\mathbf{Q}) \cap hL_\beta L_\infty^\circ h^{-1} = Z_G(\mathbf{Q}) \cap L_\beta L_\infty^\circ \text{ for all } h \in H(\mathbf{A}). \tag{4.2}$$

By (4.2), X_β is a real manifold [38, (21)]. Changing $\mathfrak{m}$ will scale all our constructions of p-adic L-functions by a fixed non-zero rational scalar (captured in the volume constant $\gamma_{p\mathfrak{m}}$ of Theorem 5.22 below); but each construction is only well-defined up to scaling the choice of periods Ω_π^ϵ, so changing $\mathfrak{m}$ yields no loss in generality. We fix $\mathfrak{m}$ to be the minimal such choice, dropping it from all notation.

By definition of $L_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}$ and (4.1), there is a proper map (see [6, Lemma 2.7])

$$\iota_\beta : X_\beta \longrightarrow S_K, \qquad [h] \longmapsto [\iota(h)\xi t_p^\beta]. \tag{4.3}$$

The cycle X_β decomposes into connected components [38, (22)] indexed by

$$\pi_0(X_\beta) := \mathscr{C}\ell_F^+(p^\beta\mathfrak{m}) \times \mathscr{C}\ell_F^+(\mathfrak{m}), \tag{4.4}$$

where the component of $[(h_1, h_2)] \in S_K$ is given by the class

$$[(\det(h_1)/\det(h_2), \det(h_2))] \in \pi_0(X_\beta).$$

For $\delta \in H(\mathbf{A}_f)$, we write $[\delta]$ for its associated class in $\pi_0(X_\beta)$ and denote the corresponding connected component

$$X_\beta[\delta] := H(\mathbf{Q})\backslash H(\mathbf{Q})\delta L_\beta H_\infty^\circ / L_\beta L_\infty^\circ.$$

Remark 4.3. We give some motivation for these definitions.

First, note that we must use the *coverings* of locally symmetric spaces for H – not the locally symmetric spaces themselves – as the map $\iota : H \to G$ does *not* induce a well-defined map on the true locally symmetric spaces. Indeed, the image of the centre $Z_H^\circ(\mathbf{R})$ under ι does not land in $Z_G^\circ(\mathbf{R})$; hence the need to quotient instead by $(Z_G \cap H)^\circ(\mathbf{R})$. Also, unlike the true locally symmetric spaces, this covering also has the correct dimension t, making the 'magic numerology' work for defining evaluation maps.

Second, the twisting matrix ξ is of paramount importance to our constructions. It will be essential in proving compatibility of evaluation maps as we vary the level β, and thus to p-adic interpolation (most notably in Proposition 6.12, via Lemma 6.2). Moreover, it plays a crucial role in the evaluation of local zeta integrals at p (see Proposition 5.12 [38, Prop. 3.4]). Whilst we do not make this explicit, underlying our later proof of p-adic interpolation is the following fact: ξ is an open orbit representative for the spherical pair $H \subset G$, in the sense that $B^-(\mathbf{Z}_p)\xi H(\mathbf{Z}_p)$ is Zariski-dense in $G(\mathbf{Z}_p)$.

4.2. Abstract evaluation maps. We axiomatise the evaluation maps of [38]. Let $K \subset G(\mathbf{A}_f)$ be open compact such that $N_Q(\mathcal{O}_\mathfrak{p}) \subset K_\mathfrak{p} \subset J_\mathfrak{p}$ for $\mathfrak{p}|p$, and recall Δ_p from §3.3. Let M be a left Δ_p-module, with action denoted $*$. Then K acts on M via its projection to $K_p \subset \Delta_p$, giving a local system $\mathscr{M}$ on S_K via §2.3.2.

4.2.1. Pulling back to cycles. We first pull back under ι_β. As in [38, §2.2.2], there is a twisting map of local systems

$$\begin{aligned}\tau_\beta^\circ : \iota_\beta^*\mathscr{M} &\longrightarrow \iota^*\mathscr{M}\\ (h, m) &\longmapsto (h, \xi t_p^\beta * m)\end{aligned}$$

where $\iota^*\mathscr{M}$ is the local system given by locally constant sections of

$$H(\mathbf{Q})\backslash(H(\mathbf{A}) \times M)/L_\beta L_\infty^\circ \to X_\beta, \qquad \zeta(h, m)\ell z = (\zeta h \ell z, \ell^{-1} * m). \tag{4.5}$$

On cohomology we get a map

$$\tau_\beta^\circ \circ \iota_\beta^* : \mathrm{H}_c^t(S_K, \mathscr{M}) \longrightarrow \mathrm{H}_c^t(X_\beta, \iota_\beta^*\mathscr{M}) \longrightarrow \mathrm{H}_c^t(X_\beta, \iota^*\mathscr{M}).$$

4.2.2. Passing to components. We trivialise $\iota^*\mathscr{M}$ by passing to connected components. Let $\delta \in H(\mathbf{A}_f)$ represent $[\delta] \in \pi_0(X_\beta)$. Define $\mathcal{X}_H := H_\infty^\circ/L_\infty^\circ$. The congruence subgroup

$$\begin{aligned}\Gamma_{\beta,\delta} &:= H(\mathbf{Q}) \cap \delta L_\beta H_\infty^\circ \delta^{-1}\\ &\subset H(\mathbf{Q})^+ := \{(h_1, h_2) \in H(\mathbf{Q}) : \det(h_i) \in \mathcal{O}_F^\times \cap F_\infty^{\times\circ}\}\end{aligned} \tag{4.6}$$

acts on $\mathcal{X}_H$ (by left translation by its embedding into H_∞°). Further if $\gamma \in \Gamma_{\beta,\delta}$ then $(\delta^{-1}\gamma\delta)_f \in L_\beta$, so $\Gamma_{\beta,\delta}$ acts on M via

$$\gamma *_{\Gamma_{\beta,\delta}} m := (\delta^{-1}\gamma\delta)_f * m. \tag{4.7}$$

If $[h_\infty] \in \mathcal{X}_H$, write $[h_\infty]_\delta$ for its image in $\Gamma_{\beta,\delta}\backslash\mathcal{X}_H$. We have a map

$$\begin{aligned} c_\delta : H_\infty^\circ &\longrightarrow H(\mathbf{A}) \\ h_\infty &\longmapsto \delta h_\infty, \end{aligned}$$

which induces a map

$$\begin{aligned} c_\delta : \Gamma_{\beta,\delta}\backslash\mathcal{X}_H &\xrightarrow{\sim} X_\beta[\delta] \subset X_\beta, \\ [h_\infty]_\delta &\longmapsto [\delta h_\infty], \end{aligned}$$

also denoted c_δ. Pulling back gives a map of local systems $c_\delta^* : \iota^*\mathscr{M} \to c_\delta^*\iota^*\mathscr{M}$.

Lemma 4.4. *The local system $c_\delta^*\iota^*\mathscr{M}$ on $\Gamma_{\beta,\delta}\backslash\mathcal{X}_H$ is given by locally constant sections of*

$$\Gamma_{\beta,\delta}\backslash[\mathcal{X}_H \times M] \longrightarrow \Gamma_{\beta,\delta}\backslash\mathcal{X}_H, \tag{4.8}$$

with action $\gamma([h_\infty], m) = ([\gamma_\infty h_\infty], \gamma *_{\Gamma_{\beta,\delta}} m)$. The map c_δ^* of local systems is induced by the map

$$\begin{aligned} c_\delta^* : H(\mathbf{Q})\backslash\big[H(\mathbf{Q})\delta L_\beta H_\infty^\circ \times M\big]/L_\beta L_\infty^\circ &\longrightarrow \Gamma_{\beta,\delta}\backslash[\mathcal{X}_H \times M], \\ (\zeta\delta\ell h_\infty, m) &\longmapsto ([h_\infty], \ell * m). \end{aligned}$$

Proof. In $H(\mathbf{Q})\backslash[H(\mathbf{Q})\delta L_\beta H_\infty^\circ \times M]/L_\beta L_\infty^\circ$, we have

$$(\zeta\delta\ell h_\infty, m) = \zeta(\delta h_\infty, \ell * m)\ell = (c_\delta(h_\infty), \ell * m),$$

so the map is as claimed. To see $c_\delta^*\iota^*\mathscr{M}$ is given by (4.8), let $\gamma \in \Gamma_{\beta,\delta}$; then $\gamma_f = \delta\ell\delta^{-1}$ for some $\ell \in L_\beta$, and $\gamma *_{\Gamma_{\beta,\delta}} m = \ell * m$ by definition. In $H(\mathbf{Q})\backslash[H(\mathbf{Q})\delta L_\beta H_\infty^\circ \times M]/L_\beta L_\infty^\circ$ we have

$$\gamma(\delta h_\infty, m) = (\gamma_f\gamma_\infty\delta h_\infty, m) = (\delta\ell\cdot\gamma_\infty h_\infty, m) = (\delta\gamma_\infty h_\infty, \ell * m).$$

Thus

$$\begin{array}{ccc} \gamma([h_\infty], m) & \xleftarrow{\quad c_\delta^* \quad} & \gamma(\delta h_\infty, m) \\ \| & & \| \\ ([\gamma_\infty h_\infty], \gamma *_{\Gamma_{\beta,\delta}} m) & \xleftarrow{c_\delta^*} & (\delta\gamma_\infty h_\infty, \gamma *_{\Gamma_{\beta,\delta}} m) \end{array}$$

commutes, from which we deduce the action must be by (4.8). ☐

4.2.3. Trivialising and integration over a fundamental class. Let

$$M_{\Gamma_{\beta,\delta}} := M/\{m - \gamma *_{\Gamma_{\beta,\delta}} m : m \in M, \gamma \in \Gamma_{\beta,\delta}\}$$

be the coinvariants of M by $\Gamma_{\beta,\delta}$. Since $\Gamma_{\beta,\delta}$ acts trivially on $M_{\Gamma_{\beta,\delta}}$, the quotient $M \to M_{\Gamma_{\beta,\delta}}$, $m \mapsto (m)_\delta$ induces a trivialisation map (over $\Gamma_{\beta,\delta}\backslash\mathcal{X}_H$)

$$\begin{aligned} \mathrm{coinv}_{\beta,\delta} : \Gamma_{\beta,\delta}\backslash[\mathcal{X}_H \times M] &\longrightarrow [\Gamma_{\beta,\delta}\backslash\mathcal{X}_H] \times M_{\Gamma_{\beta,\delta}}, \\ ([h_\infty], m) &\longmapsto ([h_\infty]_\delta, (m)_\delta), \end{aligned}$$

and thus a map from $c_\delta^*\iota^*\mathscr{M}$ to the trivial local system attached to $M_{\Gamma_{\beta,\delta}}$ on $\Gamma_{\beta,\delta}\backslash\mathcal{X}_H$. We get

$$\mathrm{coinv}_{\beta,\delta} : \mathrm{H}_c^t(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, c_\delta^*\iota^*\mathscr{M}) \to \mathrm{H}_c^t(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \otimes M_{\Gamma_{\beta,\delta}}.$$

Finally, we integrate over a fundamental class in the Borel–Moore homology $\mathrm{H}_t^{\mathrm{BM}}(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z})$. In [38, §2.2.5], a class

$$\theta_{[\delta]} \in \mathrm{H}_t^{\mathrm{BM}}(X_\beta[\delta], \mathbf{Z}) \cong \mathbf{Z}$$

is chosen for each class $[\delta]$, and we take

$$\theta_\delta := c_\delta^*(\theta_{[\delta]}).$$

Cap product

$$(-\cap\theta_\delta) : \mathrm{H}^t_{\mathrm{c}}(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \xrightarrow{\sim} \mathbf{Z}$$

induces an isomorphism

$$(-\cap\theta_\delta) : \mathrm{H}^t_{\mathrm{c}}(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \otimes M_{\Gamma_{\beta,\delta}} \xrightarrow{\sim} M_{\Gamma_{\beta,\delta}}, \qquad (\phi, m) \longmapsto (\phi\cap\theta_\delta, m).$$

Definition 4.5. Define the *evaluation map of level* p^β to be the composition

$$\begin{aligned}\mathrm{Ev}^M_{\beta,\delta} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) \xrightarrow{\tau_\beta^\circ\circ\iota_\beta^*} \mathrm{H}^t_{\mathrm{c}}(X_\beta, \iota^*\mathscr{M}) \xrightarrow{c_\delta^*} \mathrm{H}^t_{\mathrm{c}}(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, c_\delta^*\iota^*\mathscr{M}) \\ \xrightarrow{\mathrm{coinv}_{\beta,\delta}} \mathrm{H}^t_{\mathrm{c}}(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \otimes M_{\Gamma_{\beta,\delta}} \xrightarrow[\sim]{-\cap\theta_\delta} M_{\Gamma_{\beta,\delta}}.\end{aligned} \tag{4.9}$$

4.3. Variation of M, δ and β.

4.3.1. Variation in M. The functoriality in M is the object of the following statement.

Lemma 4.6. *Let $\kappa : M \to N$ be a map of Δ_p-modules. There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) & \xrightarrow{\mathrm{Ev}^M_{\beta,\delta}} & M_{\Gamma_{\beta,\delta}} \\ \downarrow \kappa & & \downarrow \kappa \\ \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{N}) & \xrightarrow{\mathrm{Ev}^N_{\beta,\delta}} & N_{\Gamma_{\beta,\delta}}. \end{array}$$

Proof. Writing out the definitions, it is immediate that κ induces a map on cohomology and a map on coinvariants, and κ commutes with each of the maps in (4.9) (compare [11, Lem. 3.2]). □

4.3.2. Variation in δ. We now investigate the dependence of the map $\mathrm{Ev}^M_{\beta,\delta}$ on the choice of δ representing $[\delta]$. The main result of §4.3.2 is Proposition 4.9, where we define modified evaluation maps independent of this choice in a special case.

For fixed δ, the action of $\ell \in L_\beta$ need not preserve $M_{\Gamma_{\beta,\delta}}$. Nevertheless:

Lemma 4.7. *Let $\delta \in H(\mathbf{A})$ and $\ell \in L_\beta$. If $\delta' = \zeta\delta\ell h_\infty \in H(\mathbf{Q})\delta\ell H^\circ_\infty \cap H(\mathbf{A}_f)$ is another representative of δ, then:*

(i) The action of ℓ on M induces a map

$$\begin{aligned} M_{\Gamma_{\beta,\delta'}} &\longrightarrow M_{\Gamma_{\beta,\delta}}, \\ (m)_{\delta'} &\longmapsto \ell * (m)_{\delta'} := (\ell * m)_\delta. \end{aligned}$$

(ii) There is a well-defined map

$$[\zeta_\infty^{-1}-] : \Gamma_{\beta,\delta'}\backslash\mathcal{X}_H \to \Gamma_{\beta,\delta}\backslash\mathcal{X}_H$$

induced by

$$[h_\infty]_{\delta'} \mapsto [\zeta_\infty^{-1}h_\infty]_\delta.$$

Proof. (i) It suffices to check $\{m' - \gamma' *_{\Gamma_{\beta,\delta'}} m' : m' \in M, \gamma' \in \Gamma_{\beta,\delta'}\}$ is mapped by $\ell * -$ to $\{m - \gamma *_{\Gamma_{\beta,\delta}} m : m \in M, \gamma \in \Gamma_{\beta,\delta}\}$. If $\gamma' \in \Gamma_{\beta,\delta'}$ and $m' \in M$, then

$$\begin{aligned}\ell * [m' - \gamma' *_{\Gamma_{\beta,\delta'}} m'] &= [\ell * m'] - [(\ell(\delta')^{-1}\gamma'\delta')_f * m'] \\ &= [\ell * m'] - (h_\infty^{-1}\delta^{-1}\zeta^{-1}\gamma'\zeta\delta h_\infty)_f * [\ell * m'] \\ &= m - (\delta^{-1}\gamma\delta)_f * m = m - \gamma *_{\Gamma_{\beta,\delta}} m,\end{aligned}$$

where $m := \ell * m' \in M$ and $\gamma := \zeta^{-1}\gamma'\zeta$. Note that $\gamma \in H(\mathbf{Q})$, as $\zeta, \gamma' \in H(\mathbf{Q})$; that γ_∞ is in H_∞°, since it has positive determinant; and that $\gamma_f \in \delta L_\beta \delta^{-1}$ by construction. Thus $\gamma \in \Gamma_{\beta,\delta}$.

(ii) The computation above shows

$$\zeta^{-1}\Gamma_{\beta,\delta'}\zeta = \Gamma_{\beta,\delta}.$$

If $[h_\infty]_{\delta'} = [h'_\infty]_{\delta'}$ then $[h_\infty] = \gamma'[h'_\infty]$ for some $\gamma' \in \Gamma_{\beta,\delta'}$; then

$$[\zeta_\infty^{-1}h_\infty] = \zeta^{-1}\gamma'[h'_\infty] = (\zeta^{-1}\gamma'\zeta)[\zeta_\infty^{-1}h'_\infty],$$

so

$$[\zeta_\infty^{-1}h_\infty]_\delta = [\zeta_\infty^{-1}h'_\infty]_\delta. \qquad \square$$

Combining Lemma 4.4 with coinvariants, the composed map of local systems is induced by

$$\begin{aligned}\mathrm{coinv}_{\beta,\delta} \circ c_\delta^* : H(\mathbf{Q})\backslash\big[H(\mathbf{Q})\delta L_\beta H_\infty^\circ \times M\big]/L_\beta L_\infty^\circ &\longrightarrow \Gamma_{\beta,\delta}\backslash\mathcal{X}_H \times M_{\Gamma_{\beta,\delta}}, \\ (\zeta\delta\ell h_\infty, m) &\longmapsto ([h_\infty]_\delta,\ (\ell * m)_\delta).\end{aligned} \tag{4.10}$$

Lemma 4.8. *Let $\delta' \in H(\mathbf{Q})\delta\ell H_\infty^\circ \cap H(\mathbf{A}_f)$ be another representative of $[\delta]$, with $\ell \in L_\beta$. Then for any class ϕ, we have*

$$\ell * \mathrm{Ev}_{\beta,\delta'}^M(\phi) = \mathrm{Ev}_{\beta,\delta}^M(\phi) \ \in M_{\Gamma_{\beta,\delta}}.$$

Proof. Write $\delta' = \zeta\delta\ell h_\infty$ with $\zeta \in H(\mathbf{Q}), h_\infty \in H_\infty^\circ$. By Lemma 4.7, we may define a map

$$([\zeta_\infty^{-1}-] \times [\ell * -]) : \Gamma_{\beta,\delta'}\backslash\mathcal{X}_H \times M_{\Gamma_{\beta,\delta'}} \longrightarrow \Gamma_{\beta,\delta}\backslash\mathcal{X}_H \times M_{\Gamma_{\beta,\delta}}$$

given by $([h_\infty]_{\delta'}, (m)_{\delta'}) \mapsto ([\zeta_\infty^{-1}h_\infty]_\delta, (\ell * m)_\delta)$. We claim there is an equality of maps

$$([\zeta_\infty^{-1}-] \times [\ell * -]) \circ [\mathrm{coinv}_{\beta,\delta'} \circ c_{\delta'}^*] = [\mathrm{coinv}_{\beta,\delta} \circ c_\delta^*]. \tag{4.11}$$

To see this, note that as δ and δ' are both trivial at infinity, $h_\infty = \zeta_\infty^{-1}$, so $[h_\infty h'_\infty]_\delta = [\zeta_\infty^{-1}-]([h'_\infty]_{\delta'})$ for all $h'_\infty \in H_\infty^\circ$. Then (4.11) follows from commutativity of

$$\begin{array}{ccccc} (\gamma'\delta'\ell' h'_\infty, m) & \xmapsto{\mathrm{coinv}_{\beta,\delta'}\circ c_{\delta'}^*} & ([h'_\infty]_{\delta'}, (\ell' * m)_{\delta'}) & \in & \Gamma_{\beta,\delta'}\backslash\mathcal{X}_H \times M_{\Gamma_{\beta,\delta'}} \\ \big\downarrow \mathrm{id} & & \big\downarrow [\zeta_\infty^{-1}-]\times[\ell*-] & & \big\downarrow [\zeta_\infty^{-1}-]\times[\ell*-] \\ (\gamma'\zeta\delta\ell\ell' h_\infty h'_\infty, m) & \xmapsto{\mathrm{coinv}_{\beta,\delta}\circ c_\delta^*} & ([h_\infty h'_\infty]_\delta, (\ell\ell' * m)_\delta) & \in & \Gamma_{\beta,\delta}\backslash\mathcal{X}_H \times M_{\Gamma_{\beta,\delta}} \end{array}$$

Note pullback by $[\zeta_\infty^{-1}-]$ induces an isomorphism

$$\mathrm{H}_t^{\mathrm{BM}}(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \xrightarrow{\sim} \mathrm{H}_t^{\mathrm{BM}}(\Gamma_{\beta,\delta'}\backslash\mathcal{X}_H, \mathbf{Z})$$

that by definition sends θ_δ to $\theta_{\delta'}$. In particular, on cohomology we get a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_c^t(\Gamma_{\beta,\delta'}\backslash\mathcal{X}_H, \mathbf{Z}) & \xrightarrow[\sim]{[\zeta_\infty^{-1}-]_*} & \mathrm{H}_c^t(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \\ & {\scriptstyle -\cap\theta_{\delta'}}\searrow^{\sim} \quad {}^{\sim}\swarrow{\scriptstyle -\cap\theta_\delta} & \\ & \mathbf{Z} & \end{array}. \tag{4.12}$$

Then we compute that

$$\begin{aligned}\ell * [\mathrm{Ev}^M_{\beta,\delta'}(\phi)] &= \ell * [(- \cap \theta_{\delta'}) \circ \mathrm{coinv}_{\beta,\delta'} \circ c^*_{\delta'} \circ \tau^\circ_\beta \circ \iota^*_\beta(\phi)] \\ &= (- \cap \theta_\delta) \circ ([\zeta_\infty^{-1} -]_* \times [\ell * -]) \circ \mathrm{coinv}_{\beta,\delta'} \circ c^*_{\delta'} \circ \tau^\circ_\beta \circ \iota^*_\beta(\phi)] \\ &= (- \cap \theta_\delta) \circ \mathrm{coinv}_{\beta,\delta} \circ c^*_\delta \circ \tau^\circ_\beta \circ \iota^*_\beta(\phi) = \mathrm{Ev}^M_{\beta,\delta}(\phi),\end{aligned}$$

where the second equality is (4.12) and the third is (4.11) combined with (4.10). □

Proposition 4.9. *Let N be a left $H(\mathbf{A})$-module, with action denoted $*$, such that $H(\mathbf{Q})$ and H°_∞ act trivially. Let $\kappa : M \to N$ be a map of L_β-modules (with N an L_β-module by restriction). Then*

$$\mathrm{Ev}^{M,\kappa}_{\beta,[\delta]} := \delta * \left[\kappa \circ \mathrm{Ev}^M_{\beta,\delta}\right] : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) \longrightarrow N$$

is well-defined and independent of the representative δ of $[\delta]$.

Proof. As $\Gamma_{\beta,\delta} \subset H(\mathbf{Q})$ acts trivially on N, κ factors through $M \twoheadrightarrow M_{\Gamma_{\beta,\delta}} \to N$, so $\kappa \circ \mathrm{Ev}^M_{\beta,\delta}$ (hence $\mathrm{Ev}^{M,\kappa}_{\beta,[\delta]}$) is well-defined. If

$$\delta' = \zeta\delta\ell h_\infty \in H(\mathbf{Q})\delta\ell H^\circ_\infty \cap H(\mathbf{A}_f),$$

then

$$\begin{aligned}\mathrm{Ev}^{M,\kappa}_{\beta,[\delta']} = \delta' * \left(\kappa \circ \mathrm{Ev}^M_{\beta,\delta'}\right) &= \delta\ell * \left(\kappa \circ \mathrm{Ev}^M_{\beta,\delta'}\right) \\ &= \delta\ell * \left(\kappa \circ \left[\ell^{-1} * \mathrm{Ev}^M_{\beta,\delta}\right]\right) = \delta * \left(\kappa \circ \mathrm{Ev}^M_{\beta,\delta}\right) = \mathrm{Ev}^{M,\kappa}_{\beta,[\delta]}.\end{aligned}$$

In the second equality we use that ζ and h_∞ act trivially on N, and the third is Lemma 4.8. □

4.3.3. Variation in β. We now investigate how evaluation maps behave as $\beta = (\beta_{\mathfrak{q}})_{\mathfrak{q}|p}$ varies. Fix $\mathfrak{p}|p$, and define $\beta' = (\beta'_{\mathfrak{q}})_{\mathfrak{q}|p}$, where $\beta'_{\mathfrak{p}} = \beta_{\mathfrak{p}} + 1$ and $\beta'_{\mathfrak{q}} = \beta_{\mathfrak{q}}$ for $\mathfrak{q} \neq \mathfrak{p}$. We have a natural projection

$$\mathrm{pr}_{\beta,\mathfrak{p}} : X_{\beta'} \longrightarrow X_\beta,$$

inducing a projection

$$\mathrm{pr}_{\beta,\mathfrak{p}} : \pi_0(X_{\beta'}) \to \pi_0(X_\beta).$$

Fix $\delta \in H(\mathbf{A}_f)$ and a set of representatives $D \subset H(\mathbf{A}_f)$ of the set $\mathrm{pr}^{-1}_{\beta,\mathfrak{p}}([\delta]) \subset \pi_0(X_{\beta'})$. For each $\eta \in D$ there exists $\ell_\eta \in L_\beta$ such that $\eta \in H(\mathbf{Q})\delta\ell_\eta H^\circ_\infty$. Via calculations directly analogous to those of Lemma 4.7, there is a map

$$M_{\Gamma_{\beta',\eta}} \to M_{\Gamma_{\beta,\delta}}, \quad (m)_\eta \mapsto \ell_\eta * (m)_\eta := (\ell_\eta * m)_\delta.$$

The action of $t_{\mathfrak{p}} \in \Delta_p$ yields an action of $U^\circ_{\mathfrak{p}}$ on $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M})$. Then we have the following direct generalisation of [38, Thm. 2.2] to general coefficients (cf. [11, Prop. 3.9]):

Proposition 4.10. *In the notation of the previous paragraph:*

(i) For each class $\Phi \in \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M})$, we have

$$\left[\mathrm{Ev}^M_{\beta,\delta} \circ U^\circ_{\mathfrak{p}}\right](\Phi) = \sum_{\eta \in D} \ell_\eta * \mathrm{Ev}^M_{\beta',\eta}(\Phi).$$

(ii) Let N and κ be as in Proposition 4.9. If $\beta_{\mathfrak{p}} \geqslant 1$, then as maps $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) \to N$ we have

$$\sum_{[\eta] \in \mathrm{pr}^{-1}_{\beta,\mathfrak{p}}([\delta])} \mathrm{Ev}^{M,\kappa}_{\beta',[\eta]} = \mathrm{Ev}^{M,\kappa}_{\beta,[\delta]} \circ U^\circ_{\mathfrak{p}}.$$

Proof. We follow closely the proof of [11, Proposition 3.4]. We recall the definition of $U_{\mathfrak{p}}^{\circ}$ from §2.3.3 (and fix simpler notation in this case). Recall $t_{\mathfrak{p}} = \iota(\varpi_{\mathfrak{p}} I_n, I_n)$, considered in $G(\mathbf{A}_f)$ by taking 1 in the components outside $\mathfrak{p}$. Let

$$K_0(\mathfrak{p}) = K \cap t_{\mathfrak{p}}^{-1} K t_{\mathfrak{p}} \qquad \text{and} \qquad K^0(\mathfrak{p}) = t_{\mathfrak{p}} K t_{\mathfrak{p}}^{-1} \cap K,$$

and denote the corresponding projections

$$\mathrm{pr}_1 : S_{K_0(\mathfrak{p})} \to S_K \qquad \text{and} \qquad \mathrm{pr}_2 : S_{K^0(\mathfrak{p})} \to S_K.$$

The action of $t_{\mathfrak{p}}$ on M induces a morphism

$$[t_{\mathfrak{p}}] : \mathrm{H}^t_{\mathrm{c}}(S_{K_0(\mathfrak{p})}, \mathscr{M}) \to \mathrm{H}^t_{\mathrm{c}}(S_{K^0(\mathfrak{p})}, \mathscr{M}),$$

induced by the map $(g, m) \mapsto (g t_{\mathfrak{p}}^{-1}, t_{\mathfrak{p}} * m)$ on local systems, and then by definition we have

$$U_{\mathfrak{p}}^{\circ} = \mathrm{Tr}(\mathrm{pr}_2) \circ [t_{\mathfrak{p}}] \circ \mathrm{pr}_1^* : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) \to \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}).$$

We give an analogue over automorphic cycles. Following the definition of $U_{\mathfrak{p}}^{\circ}$ we introduce maps

$$\begin{aligned} \iota_{\beta}^0 : X_{\beta'} &\longrightarrow S_{K^0(\mathfrak{p})}, \qquad [h] \longmapsto [\iota(h)\xi t_p^{\beta}], \\ \iota_{\beta',0} : X_{\beta'} &\longrightarrow S_{K_0(\mathfrak{p})}, \qquad [h] \longmapsto [\iota(h)\xi t_p^{\beta'}], \end{aligned}$$

which by definition fit into a commutative diagram

$$\begin{tikzcd} S_K & S_{K^0(\mathfrak{p})} \arrow[l] & S_{K_0(\mathfrak{p})} \arrow[l, "\cdot\, t_{\mathfrak{p}}^{-1}"'] \arrow[r] & S_K \\ X_{\beta} \arrow[u, "\iota_{\beta}"] & & X_{\beta'} \arrow[ll, "\mathrm{pr}_{\beta,\mathfrak{p}}"] \arrow[ul, "\iota_{\beta}^0"] \arrow[u, "\iota_{\beta',0}"'] \arrow[ur, "\iota_{\beta'}"'] & \end{tikzcd} \tag{4.13}$$

Note that the left-hand quadrilateral is Cartesian.

The action of $t_{\mathfrak{p}}$ on M induces a morphism $\iota_{\beta',0}^* \mathscr{M} \to (\iota_{\beta}^0)^* \mathscr{M}$ of sheaves over $X_{\beta'}$, giving a map

$$[t_{\mathfrak{p}}] : \mathrm{H}^t_{\mathrm{c}}(X_{\beta'}, \iota_{\beta',0}^* \mathscr{M}) \to \mathrm{H}^t_{\mathrm{c}}(X_{\beta'}, (\iota_{\beta}^0)^* \mathscr{M}).$$

Now define the analogue of $U_{\mathfrak{p}}^{\circ}$ on the cohomology of the automorphic cycles by

$$U_{\mathfrak{p}}^{\circ} = \mathrm{Tr}(\mathrm{pr}_{\beta,\mathfrak{p}}) \circ [t_{\mathfrak{p}}] : \mathrm{H}^t_{\mathrm{c}}(X_{\beta'}, \iota_{\beta'}^* \mathscr{M}) = \mathrm{H}^t_{\mathrm{c}}(X_{\beta'}, \iota_{\beta',0}^* \mathscr{M}) \to \mathrm{H}^t_{\mathrm{c}}(X_{\beta}, \iota_{\beta}^* \mathscr{M})$$

From (4.13), the definition of $U_{\mathfrak{p}}^{\circ}$, and the fact that $\beta_{\mathfrak{p}} > 0$, we get another commutative diagram

$$\begin{CD} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) @>{U_{\mathfrak{p}}^{\circ}}>> \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{M}) \\ @V{\iota_{\beta'}^*}VV @VV{\iota_{\beta}^*}V \\ \mathrm{H}^t_{\mathrm{c}}(X_{\beta'}, \iota_{\beta'}^* \mathscr{M}) @>{U_{\mathfrak{p}}^{\circ}}>> \mathrm{H}^t_{\mathrm{c}}(X_{\beta}, \iota_{\beta}^* \mathscr{M}). \end{CD}$$

Tracing back each step of the construction of the evaluation maps, and using Lemma 4.8 in the bottom square, we obtain the following commutative diagram, completing the proof of (i):

$$\begin{array}{ccc}
\mathrm{H}^t_{\mathrm{c}}(X_{\beta'}, \iota^*_{\beta'}\mathscr{M}) & \xrightarrow{\quad U_{\mathfrak{p}}^{\circ}\quad} & \mathrm{H}^t_{\mathrm{c}}(X_{\beta}, \iota^*_{\beta}\mathscr{M}) \\
\big\downarrow{\scriptstyle \tau^{\circ}_{\beta'}} & & \big\downarrow{\scriptstyle \tau^{\circ}_{\beta}} \\
\mathrm{H}^t_{\mathrm{c}}(X_{\beta'}, \iota^*\mathscr{M}) & \xrightarrow{\quad \mathrm{Tr}(\mathrm{pr}_{\beta,\mathfrak{p}})\quad} & \mathrm{H}^t_{\mathrm{c}}(X_{\beta}, \iota^*\mathscr{M}) \\
\big\downarrow{\scriptstyle \oplus_\eta c^*_\eta} & & \big\downarrow{\scriptstyle c^*_\delta} \\
\oplus_{\eta\in D}\mathrm{H}^t_{\mathrm{c}}(X_{\beta'}[\eta], c^*_\eta\iota^*\mathscr{M}) & \xrightarrow{\quad \mathrm{Tr}(\mathrm{pr}_{\beta,\mathfrak{p}})\quad} & \mathrm{H}^t_{\mathrm{c}}(X_{\beta}[\delta], c^*_\delta\iota^*\mathscr{M}) \\
{\scriptstyle \oplus_\eta(-\cap\theta_\eta)\circ\mathrm{coinv}_{\beta',\eta}}\big\downarrow & & \big\downarrow{\scriptstyle (-\cap\theta_\delta)\circ\mathrm{coinv}_{\beta,\delta}} \\
\oplus_{\eta\in D} M_{\Gamma_{\beta,\eta}} & \xrightarrow{\quad \sum_{\eta\in D}(\ell_\eta * -)\quad} & M_{\Gamma_{\beta,\delta}}.
\end{array}$$

Finally (ii) follows from (i) directly following the proof of Proposition 4.9. □

5. Classical evaluation maps and L-values

Let $K \subset G(\mathbf{A}_f)$ be an open compact subgroup as in §4.2. Now we take $M = V_\lambda^\vee(L)$, with Δ_p acting via the $*$-action defined in §3.3. We now rephrase the classical evaluation maps

$$\mathcal{E}^{j,\mathsf{w}}_{\beta,\delta} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L)) \to L \subset \overline{\mathbf{Q}}_p$$

of [38, §2.2] in the language of §4. We give two main applications of these classical evaluation maps: firstly, they provide a criterion for the existence of a Shalika model (Proposition 5.15); and when such a model exists and $K = K(\tilde{\pi})$, they compute classical L-values (Theorem 5.22).

We use an opposite convention to [38]. They take π to have weight $\lambda^\vee$ and use coefficients in V_λ. Our choices mean we replace w from *ibid.* with $-\mathsf{w}$, and $\mu^\vee$ *ibid.* with λ.

5.1. Classical evaluation maps. We recap [38, §2.2]. The *p-adic cyclotomic character* is

$$\begin{aligned}
\chi_{\mathrm{cyc}} : F^\times\backslash\mathbf{A}_F^\times &\longrightarrow \mathbf{Z}_p^\times \\
y &\longmapsto \mathrm{sgn}(y_\infty)\cdot|y_f|\cdot \mathrm{N}_{F/\mathbf{Q}}(y_p) := \prod_{\sigma\in\Sigma}\mathrm{sgn}(y_\sigma)\cdot|y_f|\cdot\prod_{\mathfrak{p}|p} N_{F_{\mathfrak{p}}/\mathbf{Q}_p}(y_{\mathfrak{p}}).
\end{aligned} \tag{5.1}$$

This is the p-adic character associated to the adelic norm (e.g. [16, §2.2.2]), and is trivial on $F_\infty^{\times\circ}$.

Definition 5.1. For $(j_1, j_2) \in \mathbf{Z}^2$, let $V^H_{(j_1,j_2)}$ be a 1-dimensional L-vector space with $H(\mathbf{A})$-action

$$(h_1, h_2) * v := \chi_{\mathrm{cyc}}\big[\det(h_1)^{j_1}\det(h_2)^{j_2}\big]v,$$

for $h_1, h_2 \in \mathrm{GL}_n(\mathbf{A}_F)$ and $v \in V^H_{(j_1,j_2)}$. This is the set of L-points of the algebraic representation of H of highest weight $(j_1, ..., j_1, j_2, ..., j_2)$. Note that $H(\mathbf{Q})$ and H_∞° act trivially on $V^H_{(j_1,j_2)}$. For $(\ell_1, \ell_2) \in L_\beta$, as $|\det(\ell_i)_f| = 1$, we have

$$(\ell_1, \ell_2) * v := \mathrm{N}_{F/\mathbf{Q}}\big[\det(\ell_{1,p})^{j_1}\det(\ell_{2,p})^{j_2}\big]v.$$

The following branching law for $H \subset G$ gives a representation-theoretic description of $\mathrm{Crit}(\lambda)$.

Lemma 5.2. *Let $j \in \mathbf{Z}$. We have $j \in \mathrm{Crit}(\lambda)$ if and only if*

$$\dim_L(\mathrm{Hom}_{H(\mathbf{Z}_p)}(V_\lambda^\vee, V^H_{(j,-\mathsf{w}-j)})) = 1.$$

(Note that since the $*$- and $\cdot$-actions of $H(\mathbf{Z}_p)$ on $V_\lambda^\vee$ coincide, there is no ambiguity here).

Proof. By [45, Prop. 6.3], we know $0 \in \mathrm{Crit}(\lambda)$ if and only if

$$\dim_L(\mathrm{Hom}_{H(\mathbf{Z}_p)}(V_\lambda^\vee, V_{(0,-\mathsf{w})}^H)) = 1.$$

Note $L(\pi, j+\frac{1}{2})$ is critical if and only if $L(\pi \otimes |\cdot|^j, \frac{1}{2})$ is critical. Let

$$\tilde{\lambda} = \lambda + j(1,...,1),$$

of purity weight $\mathsf{w}+2j$; then $j \in \mathrm{Crit}(\lambda)$ if and only if $0 \in \mathrm{Crit}(\tilde{\lambda})$, and in this case

$$\begin{aligned} 1 &= \dim_L(\mathrm{Hom}_{H(\mathbf{Z}_p)}(V_{\tilde{\lambda}}^\vee, V_{(0,-\mathsf{w}-2j)}^H) \\ &= \dim_L(\mathrm{Hom}_{H(\mathbf{Z}_p)}(V_\lambda^\vee, V_{(j,-\mathsf{w}-j)}^H)). \end{aligned}$$

□

Recall the map

$$\tau_\beta^\circ \circ \iota_\beta^* : \mathrm{H}_c^t(S_K, \mathscr{V}_\lambda^\vee) \to \mathrm{H}_c^t(X_\beta, \iota^*\mathscr{V}_\lambda^\vee)$$

from §4.2. For $j \in \mathrm{Crit}(\lambda)$, fix a basis $\kappa_{\lambda,j}$ of $\mathrm{Hom}_{H(\mathbf{Z}_p)}(V_\lambda^\vee, V_{(j,-\mathsf{w}-j)}^H)$. This induces a homomorphism

$$\kappa_{\lambda,j} : \mathrm{H}_c^t(X_\beta, \iota^*\mathscr{V}_\lambda^\vee) \longrightarrow \mathrm{H}_c^t(X_\beta, \mathscr{V}_{(j,-\mathsf{w}-j)}^H),$$

where $\mathscr{V}_{(j,-\mathsf{w}-j)}^H$ is the local system defined as in §2.3.2. Let $\delta \in H(\mathbf{A}_f)$. As in §4.2, applying $(-\cap\theta_\delta)\circ \mathrm{coinv}_{\beta,\delta}\circ c_\delta^*$ and choosing a basis u_j of $V_{(j,-\mathsf{w}-j)}^H$ gives a map

$$\mathrm{H}_c^t(X_\beta, \mathscr{V}_{(j,-\mathsf{w}-j)}^H) \xrightarrow{\mathrm{coinv}_{\beta,\delta}\circ c_\delta^*} \mathrm{H}_c^t(\Gamma_{\beta,\delta}\backslash\mathcal{X}_H, \mathbf{Z}) \otimes V_{(j,-\mathsf{w}-j)}^H \xrightarrow[\sim]{(-\cap\theta_\delta)\otimes\mathrm{id}} V_{(j,-\mathsf{w}-j)}^H \cong L.$$

Then in [38, (33)], the authors define

$$\mathcal{E}_{\beta,\delta}^{j,\mathsf{w}} := (-\cap\theta_\delta)\circ\mathrm{coinv}_{\beta,\delta}\circ c_\delta^*\circ(\kappa_{\lambda,j})_*\circ\tau_\beta^\circ\circ\iota_\beta^*.$$

The choice of basis u_j of $V_{(j,-\mathsf{w}-j)}^H$ identifies $V_{(j,-\mathsf{w}-j)}^H$ with L, and we get a map $\kappa_{\lambda,j}^\circ$ of $H(\mathbf{Z}_p)$-modules defined via

$$\kappa_{\lambda,j}^\circ : V_\lambda^\vee(L) \longrightarrow L, \qquad \kappa_{\lambda,j}(\mu) = \kappa_{\lambda,j}^\circ(\mu)\cdot u_j \quad \text{for all } \mu \in V_\lambda^\vee(L). \tag{5.2}$$

As $\Gamma_{\beta,\delta}$ acts trivially on $V_{(j,-\mathsf{w}-j)}^H$, $\kappa_{\lambda,j}$ and $\kappa_{\lambda,j}^\circ$ factor through $(V_\lambda^\vee(L))_{\Gamma_{\beta,\delta}}$. It is easy to see that $\kappa_{\lambda,j}$ commutes with restricting to components, passing to coinvariants, and integrating against the fundamental class. We deduce the following description of $\mathcal{E}_{\beta,\delta}^{j,\mathsf{w}}$ via §4.2:

Lemma 5.3. *We have* $\mathcal{E}_{\beta,\delta}^{j,\mathsf{w}} = \kappa_{\lambda,j}^\circ \circ \mathrm{Ev}_{\beta,\delta}^{V_\lambda^\vee}$.

Recall from [38, (33)] the map

$$\mathcal{E}_{\beta,[\delta]}^{j,\mathsf{w}} := \delta * \mathcal{E}_{\beta,\delta}^{j,\mathsf{w}} = \chi_{\mathrm{cyc}}\big(\det(\delta_1^j\delta_2^{-\mathsf{w}-j})\big)\mathcal{E}_{\beta,\delta}^{j,\mathsf{w}},$$

is independent of the representative δ of $[\delta] \in \pi_0(X_\beta)$. This also follows from Propositions 4.9,4.10.

Recall

$$\pi_0(X_\beta) = \mathscr{C}\ell_F^+(p^\beta\mathfrak{m}) \times \mathscr{C}\ell_F^+(\mathfrak{m})$$

from (4.4). Write $\mathrm{pr}_1, \mathrm{pr}_2$ for the projections of $\pi_0(X_\beta)$ onto the first and second factors respectively, and let pr_β denote the natural composition

$$\mathrm{pr}_\beta : \mathscr{C}\ell_F^+(p^\beta\mathfrak{m}) \times \mathscr{C}\ell_F^+(\mathfrak{m}) \xrightarrow{\mathrm{pr}_1} \mathscr{C}\ell_F^+(p^\beta\mathfrak{m}) \longrightarrow \mathscr{C}\ell_F^+(p^\beta). \tag{5.3}$$

Definition 5.4. Let η_0 be any finite order character of $\mathscr{C}\ell_F^+(\mathfrak{m})$, and $\mathbf{x} \in \mathscr{C}\ell_F^+(p^\beta)$. Define an η_0-averaged evaluation map

$$\mathcal{E}_{\beta,\mathbf{x}}^{j,\eta_0} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L)) \to L$$

by

$$\mathcal{E}_{\beta,\mathbf{x}}^{j,\eta_0} := \sum_{[\delta]\in \mathrm{pr}_\beta^{-1}(\mathbf{x})} \eta_0^{-1}\big(\mathrm{pr}_2([\delta])\big)\ \mathcal{E}_{\beta,[\delta]}^{j,\mathsf{w}}.$$

In [38] this is denoted $\mathcal{E}_{\beta,\mathbf{x}}^{j,\eta}$, where $\eta = \eta_0|\cdot|^{\mathsf{w}}$; as later w will vary whilst η_0 will not, we continue to use a superscript η_0 instead of η throughout, with w implicit in the source.

Let χ be a finite order Hecke character of conductor (exactly) $p^{\beta'}$, for $\beta' = (\beta'_{\mathfrak{p}})_{\mathfrak{p}|p}$. Let $\beta_{\mathfrak{p}} = \max(\beta'_{\mathfrak{p}}, 1)$ and $\beta = (\beta_{\mathfrak{p}})_{\mathfrak{p}|p}$. Then χ induces a character on $\mathscr{C}\ell_F^+(p^\beta)$. Let $L(\chi)$ be the smallest extension of L containing $\chi(\mathscr{C}\ell_F^+(p^\beta))$. For $j \in \mathrm{Crit}(\lambda)$, define

$$\begin{aligned}\mathcal{E}_\chi^{j,\eta_0} = \sum_{\mathbf{x}\in\mathscr{C}\ell_F^+(p^\beta)} \chi(\mathbf{x})\ \mathcal{E}_{\beta,\mathbf{x}}^{j,\eta_0}\ :\ &\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee(L)) \longrightarrow L(\chi), \\ &\phi \longmapsto \sum_{[\delta]\in\pi_0(X_\beta)} \chi\big(\mathrm{pr}_\beta([\delta])\big)\cdot \eta_0^{-1}\big(\mathrm{pr}_2([\delta])\big)\cdot\Big(\delta * \Big[\kappa_{\lambda,j}^\circ \circ \mathrm{Ev}_{\beta,\delta}^{V_\lambda^\vee}(\phi)\Big]\Big).\end{aligned} \tag{5.4}$$

Remark 5.5. Summarising, $\mathcal{E}_\chi^{j,\eta_0}$ is the composition

$$\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee) \xrightarrow{\oplus \mathrm{Ev}_{\beta,\delta}^{V_\lambda^\vee}} \bigoplus_{[\delta]} (V_\lambda^\vee)_{\Gamma_{\beta,\delta}} \xrightarrow{v\mapsto \delta*\kappa_{\lambda,j}^\circ(v)} \bigoplus_{[\delta]} L \xrightarrow{\oplus \Xi_{\mathbf{x}}^{\eta_0}} \bigoplus_{\mathbf{x}} L \xrightarrow{\ell\mapsto\Sigma\chi(\mathbf{x})\ell_{\mathbf{x}}} L, \tag{5.5}$$

$$\oplus\mathcal{E}_{\beta,[\delta]}^{j,\mathsf{w}} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee) \to \bigoplus_{[\delta]} L, \qquad \oplus\mathcal{E}_{\beta,\mathbf{x}}^{j,\eta_0} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee) \to \bigoplus_{\mathbf{x}} L,$$

where the sums are over $[\delta] \in \pi_0(X_\beta)$ or $\mathbf{x} \in \mathscr{C}\ell_F^+(p^\beta)$, and $\Xi_{\mathbf{x}}^{\eta_0}$ is the η_0-averaging map

$$\Xi_{\mathbf{x}}^{\eta_0} : (m_{[\delta]})_{[\delta]} \longmapsto \sum_{[\delta]\in\mathrm{pr}_\beta^{-1}(\mathbf{x})} \eta_0^{-1}(\mathrm{pr}_2([\delta]))\cdot m_{[\delta]}.$$

5.2. Compatible choices of bases: branching laws for $H \subset G$.

Let $j \in \mathrm{Crit}(\lambda)$. The map $\mathcal{E}_\chi^{j,\eta_0}$ depends on choices of bases

$$u_j \text{ of } V^H_{(j,-\mathsf{w}-j)} \cong L \qquad \text{and} \qquad \kappa_{\lambda,j} \text{ of } \mathrm{Hom}_{H(\mathbf{Z}_p)}(V_\lambda^\vee(L), V^H_{(j,-\mathsf{w}-j)}),$$

which we combined into a single choice of non-zero $\kappa_{\lambda,j}^\circ$ in (5.2). At present, we have made a separate, independent choice for each j. For p-adic interpolation it is essential to make all these choices compatibly. We now do this via branching laws.

5.2.1. Idea: critical integers via branching laws. Dualising Lemma 5.2 gives a reinterpretation of the set $\mathrm{Crit}(\lambda)$ in terms of *branching laws* for $H \subset G$, describing characters of H that appear in $V_\lambda|_H$ with multiplicity 1. For each $j \in \mathrm{Crit}(\lambda)$, we obtain a line $V^H_{(-j,\mathsf{w}+j)} \subset V_\lambda|_H$. Our key idea for p-adic interpolation is to reinterpret this again in terms of smaller groups; instead of considering branching laws for $H \subset G$, one can consider branching laws for $G_n = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\mathrm{GL}_n \subset H$, embedded diagonally. Indeed, recall λ is pure with purity weight w, and V_λ^H is the irreducible representation of H of highest weight λ; then as G_n-representations we have

$$\begin{aligned} V_\lambda^H|_{G_n} &\cong V_{\lambda'}^{G_n} \otimes V_{\lambda''}^{G_n} \\ &\cong V_{\lambda'}^{G_n} \otimes (V_{\lambda'}^{G_n})^\vee \otimes (\mathrm{N}_{F/\mathbf{Q}}\circ\det)^{\mathsf{w}}, \end{aligned} \tag{5.6}$$

recalling $\lambda' = (\lambda_1, \dots, \lambda_n)$ and $\lambda'' = (\lambda_{n+1}, \dots, \lambda_{2n})$. As $V_{\lambda'}^{G_n} \otimes (V_{\lambda'}^{G_n})^\vee$ contains the trivial representation with multiplicity 1, $V_\lambda^H|_{G_n}$ contains $(\mathrm{N}_{F/\mathbf{Q}} \circ \det)^{\mathsf{w}}$ with multiplicity 1. In Notation 5.6 and Lemmas 5.7 and 5.8, we show that the $\#\mathrm{Crit}(\lambda)$ *different* lines $V_{(-j,\mathsf{w}+j)}^H$ in $V_\lambda|_H$ (given by Lemma 5.2) can all be collapsed onto this *single* line in $V_\lambda^H|_{G_n}$. Choosing a generator of this single line thus allows us to align generators of the distinct lines $V_{(-j,\mathsf{w}+j)}^H$ for $j \in \mathrm{Crit}(\lambda)$.

5.2.2. Passing from $H \subset G$ to $G_n \subset H$. Let $j \in \mathrm{Crit}(\lambda)$, and

$$\kappa_{\lambda,j} \in \mathrm{Hom}_{H(\mathbf{Z}_p)}(V_\lambda^\vee(L), V_{(j,-\mathsf{w}-j)}^H)$$

and

$$u_j \in V_{(j,-\mathsf{w}-j)}^H$$

be auxiliary bases. We have a dual basis

$$u_j^\vee \text{ of } V_{(-j,\mathsf{w}+j)}^H \cong (V_{(j,-\mathsf{w}-j)}^H)^\vee.$$

Dualising $\kappa_{\lambda,j}$ gives a map

$$\kappa_{\lambda,j}^\vee : V_{(-j,\mathsf{w}+j)}^H \longrightarrow (V_\lambda^\vee(L))^\vee \cong V_\lambda(L)$$

of $H(\mathbf{Z}_p)$-modules. Then $\kappa_{\lambda,j}^\vee(u_j^\vee) \in V_\lambda(L)$ generates the unique $H(\mathbf{Z}_p)$-submodule isomorphic to $V_{(-j,\mathsf{w}+j)}^H$ inside $V_\lambda(L)|_{H(\mathbf{Z}_p)}$.

Notation 5.6. Viewing $\kappa_{\lambda,j}^\vee(u_j^\vee) \in V_\lambda(L)$ as an element of $\mathrm{Ind}_{Q^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} V_\lambda^H(L)$ by Lemma 3.6, let

$$v_{\lambda,j}^H := \kappa_{\lambda,j}^\vee(u_j^\vee)\left[\left(\begin{smallmatrix} I_n & I_n \\ 0 & I_n \end{smallmatrix}\right)\right] \in V_\lambda^H(L).$$

Let

$$N_Q^\times(\mathbf{Z}_p) := \{\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in N_Q(\mathbf{Z}_p) : X \in G_n(\mathbf{Z}_p)\} \subset N_Q(\mathbf{Z}_p).$$

Lemma 5.7. *(i) For each $\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in N_Q^\times(\mathbf{Z}_p)$, we have*

$$\kappa_{\lambda,j}^\vee(u_j^\vee)\left[\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)\right] = \left[\mathrm{N}_{F/\mathbf{Q}} \circ \det(X)\right]^j \left(\langle \left(\begin{smallmatrix} X & \\ & 1 \end{smallmatrix}\right) \rangle_\lambda \cdot v_{\lambda,j}^H \right).$$

(ii) The vector $v_{\lambda,j}^H \in V_\lambda^H(L)$ is non-zero.

Proof. (i) For $\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in N_Q^\times(\mathbf{Z}_p)$, we have

$$\begin{aligned} \kappa_{\lambda,j}^\vee(u_j^\vee)\left[\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)\right] &= \kappa_{\lambda,j}^\vee(u_j^\vee)\left[\left(\begin{smallmatrix} X & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} X^{-1} & \\ & 1 \end{smallmatrix}\right)\right] \\ &= \langle \left(\begin{smallmatrix} X & \\ & 1 \end{smallmatrix}\right) \rangle_\lambda \cdot \left(\kappa_{\lambda,j}^\vee(u_j^\vee)\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} X^{-1} & \\ & 1 \end{smallmatrix}\right)\right] \right), \end{aligned}$$

where the last equality follows by (3.2). Moreover, $\left(\begin{smallmatrix} X^{-1} & \\ & 1 \end{smallmatrix}\right) \in H(\mathbf{Z}_p) \subset G(\mathbf{Z}_p)$ acts on $\kappa_{\lambda,j}^\vee(u_j^\vee)$ by right translation, and $\kappa_{\lambda,j}^\vee$ is $H(\mathbf{Z}_p)$-equivariant, whence we see

$$\begin{aligned} \kappa_{\lambda,j}^\vee(u_j^\vee)\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} X^{-1} & \\ & 1 \end{smallmatrix}\right)\right] &= \left(\left(\begin{smallmatrix} X^{-1} & \\ & 1 \end{smallmatrix}\right) \cdot \kappa_{\lambda,j}^\vee(u_j^\vee) \right)\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right] \\ &= \kappa_{\lambda,j}^\vee\left(\left(\begin{smallmatrix} X^{-1} & \\ & 1 \end{smallmatrix}\right) \cdot u_j^\vee \right)\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right] \\ &= (\mathrm{N}_{F/\mathbf{Q}} \circ \det(X))^j \cdot \kappa_{\lambda,j}^\vee(u_j^\vee)\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right] \\ &= (\mathrm{N}_{F/\mathbf{Q}} \circ \det(X))^j \cdot v_{\lambda,j}^H, \end{aligned}$$

using that $u_j^\vee \in V_{(-j,\mathsf{w}+j)}^H$. Combining these equalities proves (i).

(ii) Suppose $v^H_{\lambda,j} = 0$. By (i), we see

$$\kappa^\vee_{\lambda,j}(u^\vee_j)|_{N_Q^\times(\mathbf{Z}_p)} = 0.$$

Since $N_Q^\times(\mathbf{Z}_p)$ is Zariski-dense in $N_Q(\mathbf{Z}_p)$, we deduce $\kappa^\vee_{\lambda,j}(u^\vee_j)$ vanishes on $N_Q(\mathbf{Z}_p)$, hence on J_p by the parahoric decomposition; but by Zariski-density of $J_p \subset G(\mathbf{Z}_p)$ this forces $\kappa^\vee_{\lambda,j}(u^\vee_j) = 0$. This is absurd by its definition. □

Alternative choices of $\kappa_{\lambda,j}$ or u_j scale $v^H_{\lambda,j}$ by $L^\times$-multiple. As $v^H_{\lambda,j}$ is non-zero, we see choosing $\kappa_{\lambda,j}$ and u_j is equivalent to fixing a basis of the line $L \cdot v^H_{\lambda,j}$. This line is independent of j:

Lemma 5.8. *(i) Let $h \in G_n(\mathbf{Z}_p)$. Then*

$$\left\langle \left(\begin{smallmatrix} h & \\ & h\end{smallmatrix}\right)\right\rangle_\lambda \cdot v^H_{\lambda,j} = (\mathrm{N}_{F/\mathbf{Q}} \circ \det(h))^{\mathsf{w}}\ v^H_{\lambda,j}.$$

(ii) The line $L \cdot v^H_{\lambda,j} \subset V^H_\lambda(L)$ is independent of j.

Proof. (i) By definition: $\langle\cdot\rangle_\lambda$ acts on $\kappa^\vee_{\lambda,j}(u^\vee_j)$ by left translation; the $\cdot$-action of $H(\mathbf{Z}_p)$ on $\kappa^\vee_{\lambda,j}(u^\vee_j)$ is by right translation; and $(h_1, h_2) \in H(\mathbf{Z}_p)$ acts on $u^\vee_j$ by $\mathrm{N}_{F/\mathbf{Q}}(\det(h_1)^{-j}\det(h_2)^{\mathsf{w}+j})$. Then

$$\begin{aligned}
\left\langle \left(\begin{smallmatrix} h & \\ & h\end{smallmatrix}\right)\right\rangle_\lambda \cdot v^H_{\lambda,j} &= \kappa^\vee_{\lambda,j}(u^\vee_j)\left[\left(\begin{smallmatrix} h & \\ & h\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)\right]\\
&= \kappa^\vee_{\lambda,j}(u^\vee_j)\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} h & \\ & h\end{smallmatrix}\right)\right]\\
&= \left(\left(\begin{smallmatrix} h & \\ & h\end{smallmatrix}\right)\cdot \kappa^\vee_{\lambda,j}(u^\vee_j)\right)\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)\right]\\
&= \kappa^\vee_{\lambda,j}\left(\left(\begin{smallmatrix} h & \\ & h\end{smallmatrix}\right)\cdot u^\vee_j\right)\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)\right]\\
&= (\mathrm{N}_{F/\mathbf{Q}} \circ \det(h))^{\mathsf{w}}\kappa^\vee_{\lambda,j}(u^\vee_j)\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)\right]\\
&= (\mathrm{N}_{F/\mathbf{Q}} \circ \det(h))^{\mathsf{w}} v^H_{\lambda,j}.
\end{aligned}$$

In the first equality, we use the transformation law from (3.2) for $\kappa^\vee_{\lambda,j}(u^\vee_j) \in V_\lambda$ (via Lemma 3.6).

(ii) As after (5.6), the restriction $V^H_\lambda|_{G_n}$ contains $(\mathrm{N}_{F/\mathbf{Q}} \circ \det)^{\mathsf{w}}$ as a unique summand. This summand visibly has no dependence on j, but by (i), for each j it coincides with $L \cdot v^H_{\lambda,j}$. □

Thus evaluation at $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ collapses all the lines $V^H_{(-j,\mathsf{w}+j)} \subset V_\lambda|_H$ onto the *same* line in $V^H_\lambda|_{G_n}$.

5.2.3. From $G_n \subset H$ back to $H \subset G$. We now use §5.2.2 to align our initial choices of $\kappa^\circ_{\lambda,j}$.

Notation 5.9. Fix a generator v^H_λ of $(\mathrm{N}_{F/\mathbf{Q}} \circ \det)^{\mathsf{w}} \subset V^H_\lambda|_{G_n}$. We take

$$v^H_\lambda \in V^H_\lambda(\mathcal{O}_L)$$

optimally integrally normalised (in the sense that $\varpi_L^{-1} v^H_\lambda \notin V^H_\lambda(\mathcal{O}_L)$).

Definition 5.10. Using Lemma 5.8(ii), rescale $\kappa_{\lambda,j}$ and u_j so that

$$v^H_{\lambda,j} = (-1)^{dnj} v^H_\lambda.$$

Then let $\kappa^\circ_{\lambda,j} : V^\vee_\lambda(L) \longrightarrow L$ be the map determined by the property (5.2).

From the definitions, and using duality, we can describe $\kappa^\circ_{\lambda,j}$ as the map

$$\begin{aligned}\kappa^\circ_{\lambda,j} : V^\vee_\lambda(L) &\longrightarrow L,\\ \mu &\longmapsto \mu[\kappa^\vee_{\lambda,j}(u^\vee_j)].\end{aligned} \tag{5.7}$$

now give an alternative description of $\kappa^\circ_{\lambda,j}$ better suited to p-adic interpolation.

Lemma 5.11. *(i) For each j, there exists a unique*

$$[v_{\lambda,j} : G(\mathbf{Z}_p) \to V_\lambda^H(L)] \in V_\lambda(L)$$

with

$$v_{\lambda,j}\left[\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)\right] = (-1)^{dnj}\left[\mathrm{N}_{F/\mathbf{Q}} \circ \det(X)\right]^j \left(\langle \left(\begin{smallmatrix} X & \\ & 1 \end{smallmatrix}\right) \rangle_\lambda \cdot v_\lambda^H \right) \tag{5.8}$$

for $\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in N_Q^\times(\mathbf{Z}_p)$.

(ii) For $(h_1, h_2) \in H(\mathbf{Z}_p)$, we have

$$\left(\begin{smallmatrix} h_1 & \\ & h_2 \end{smallmatrix}\right) \cdot v_{\lambda,j} = \mathrm{N}_{F/\mathbf{Q}}[\det(h_1)^{-j} \det(h_2)^{\mathsf{w}+j}] v_{\lambda,j}.$$

(iii) The map

$$\kappa_{\lambda,j}^\circ : V_\lambda^\vee(L) \longrightarrow L$$

from Definition 5.10 is given by $\mu \mapsto \mu(v_{\lambda,j})$.

Proof. (i) We take

$$v_{\lambda,j} := \kappa_{\lambda,j}^\vee(u_j^\vee).$$

Then (5.8) is exactly Lemma 5.7(i). Note the values of $v_{\lambda,j}$ on $N_Q^-(\mathbf{Z}_p)H(\mathbf{Z}_p)N_Q^\times(\mathbf{Z}_p)$ are determined by (5.8) and the transformation property of $\mathrm{Ind}_{Q^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} V_\lambda^H(L)$; and this is Zariski-dense in $G(\mathbf{Z}_p)$. Hence $v_{\lambda,j}$ is unique with this property.

(ii) Since $\kappa_{\lambda,j}^\vee$ is $H(\mathbf{Z}_p)$-equivariant and $u_j^\vee \in V_{(-j,\mathsf{w}+j)}^H$, we compute that

$$\begin{aligned}\left(\begin{smallmatrix} h_1 & \\ & h_2 \end{smallmatrix}\right) \cdot v_{\lambda,j} &= \kappa_{\lambda,j}^\vee\left(\left(\begin{smallmatrix} h_1 & \\ & h_2 \end{smallmatrix}\right) \cdot u_j^\vee\right) \\ &= \mathrm{N}_{F/\mathbf{Q}}[\det(h_1)^{-j} \det(h_2)^{\mathsf{w}+j}] v_{\lambda,j}.\end{aligned}$$

(iii) This follows directly from (5.7). □

5.2.4. Comparison with previous work. In [38, (40)], the authors choose a lowest weight vector $v_0 \in V_\lambda^\vee(L)$, and use this choice and Lie theory to define an integral lattice

$$V_\lambda^\vee(\mathcal{O}_L)^{\mathrm{DJR}} \subset V_\lambda^\vee(L)$$

(which may be different from the lattice $V_\lambda(\mathcal{O}_L)$ defined in §2.2). For $j \in \mathrm{Crit}(\lambda)$, they construct a map

$$\kappa_j^{\mathrm{DJR}} : V_\lambda^\vee(\mathcal{O}_L)^{\mathrm{DJR}} \to V_{(j,-\mathsf{w}-j)}^H(\mathcal{O}_L) \cong \mathcal{O}_L,$$

normalised so that $\kappa_j^{\mathrm{DJR}}(\xi \cdot v_0) = 1$ (which they prove is possible in results analogous to §5.2.2). This map is denoted κ_j° *ibid.*

We freely identify κ_j^{DJR} with its scalar extension $V_\lambda^\vee(L) \to L$. By Lemma 5.2, for each $j \in \mathrm{Crit}(\lambda)$ the maps $\kappa_{\lambda,j}^\circ$ and κ_j^{DJR} agree up to $L^\times$-multiple. Fix $j_0 \in \mathrm{Crit}(\lambda)$; we can align the choice of v_0 (and hence the integral structure $V_\lambda^\vee(\mathcal{O}_L)^{\mathrm{DJR}}$) in [38] so that $\kappa_{\lambda,j_0}^\circ = \kappa_{j_0}^{\mathrm{DJR}}$. Then:

Proposition 5.12. *For each $j \in \mathrm{Crit}(\lambda)$, we have $\kappa_{\lambda,j}^\circ = \kappa_j^{\mathrm{DJR}}$.*

Proof. Dualising the map κ_j^{DJR}, and evaluating at $1 \in \mathcal{O}_L$, one obtains an element

$$v_j^{\mathrm{DJR}} \in V_\lambda(\mathcal{O}_L)^{\mathrm{DJR}} \qquad \text{such that} \qquad \kappa_j^{\mathrm{DJR}}(\mu) = \mu(v_j^{\mathrm{DJR}}).$$

Moreover

$$v_j^{\mathrm{DJR}} \in V_{(-j,\mathsf{w}+j)}^H \subset V_\lambda|_H,$$

so v_j^{DJR} is an $L^\times$-multiple of $v_{\lambda,j}$ from Lemma 5.11. In particular, there exists $c_j \in \mathcal{O}_L$ such that either $v_{\lambda,j} = c_j v_j^{\mathrm{DJR}}$ or $c_j v_{\lambda,j} = v_j^{\mathrm{DJR}}$. We assume the latter; the proof is identical for the former. By the above and Lemma 5.11(iii), it suffices to prove that $c_j = 1$ for each $j \in \mathrm{Crit}(\lambda)$. By assumption $c_{j_0} = 1$.

By [38, Prop. 2.6], for all $j \in \mathrm{Crit}(\lambda)$, $\mu \in V_\lambda^\vee(\mathcal{O}_L)$, and $\beta \in \mathbf{Z}_{\geqslant 1}$, we have

$$\mu[(\xi^{-1}t_p^\beta) * v_j^{\mathrm{DJR}}] \equiv \mu[(\xi^{-1}t_p^\beta) * v_{j_0}^{\mathrm{DJR}}] \pmod{p^\beta \mathcal{O}_L}.$$

As this holds for all μ, by considering $\mathcal{O}_L$-bases we deduce

$$(\xi^{-1}t_p^\beta) * \left[v_j^{\mathrm{DJR}} - v_{j_0}^{\mathrm{DJR}}\right] \in p^\beta V_\lambda(\mathcal{O}_L)^{\mathrm{DJR}}.$$

Any two integral lattices in $V_\lambda(L)$ are commensurable, so there exists $\beta_0 \in \mathbf{Z}_{\geqslant 0}$ such that

$$(\xi^{-1}t_p^\beta) * \left[v_j^{\mathrm{DJR}} - v_{j_0}^{\mathrm{DJR}}\right] \in p^{\beta-\beta_0} V_\lambda(\mathcal{O}_L),$$

for all $\beta \geqslant \beta_0$, and in particular, our normalisations ensure we have

$$(\xi^{-1}t_p^\beta) * \left[c_j v_{\lambda,j} - v_{\lambda,j_0}\right] \in p^{\beta-\beta_0} V_\lambda(\mathcal{O}_L). \tag{5.9}$$

Thus, considering this element in $\mathrm{Ind}_{Q^-(\mathbf{Z}_p)}^{G(\mathbf{Z}_p)} V_\lambda^H(L)$ via Lemma 3.6, for all $g \in G(\mathbf{Z}_p)$ we have

$$(\xi^{-1}t_p^\beta) * \left[c_j v_{\lambda,j} - v_{\lambda,j_0}\right](g) \in p^{\beta-\beta_0} V_\lambda^H(\mathcal{O}_L). \tag{5.10}$$

Recall $v_\lambda^H \in L[H]$ (from Notation 5.9) is polynomial in the coordinates of H; after possibly enlarging β_0, we may assume that $\varpi_L^{\beta_0} v_\lambda^H \in \mathcal{O}_L[H]$, that is, the coefficients are all integral. As the action $\langle\cdot\rangle_\lambda$ on v_λ^H is by right-translation, one deduces easily that if $h, h' \in H(\mathbf{Z}_p)$ with $h \equiv h' \pmod{p^\beta}$, then

$$\langle h\rangle_\lambda \cdot \varpi_L^{\beta_0} v_\lambda^H \equiv \langle h'\rangle_\lambda \cdot \varpi_L^{\beta_0} v_\lambda^H \pmod{p^\beta V_\lambda^H(\mathcal{O}_L)},$$

so

$$\langle h\rangle_\lambda \cdot v_\lambda^H \equiv \langle h'\rangle_\lambda \cdot v_\lambda^H \pmod{p^{\beta-\beta_0} V_\lambda^H(\mathcal{O}_L)}. \tag{5.11}$$

Now, by §3.3 note the action of $\xi^{-1}t_p^\beta$ on V_λ is induced by the action

$$\left(\begin{smallmatrix} 1 & X \\ 0 & 1\end{smallmatrix}\right) \longmapsto \left[t_p^\beta \left(\begin{smallmatrix} 1 & X \\ 0 & 1\end{smallmatrix}\right) t_p^{-\beta}\right] \xi^{-1} = \begin{pmatrix} 1 & p^\beta X \\ 0 & 1\end{pmatrix} \left(\begin{smallmatrix} 1 & -1 \\ 0 & w_n\end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & \\ & w_n\end{smallmatrix}\right) \begin{pmatrix} 1 & -1+p^\beta X w_n \\ 0 & 1\end{pmatrix} \tag{5.12}$$

on $\left(\begin{smallmatrix} 1 & X \\ 0 & 1\end{smallmatrix}\right) \in N_Q^\times(\mathbf{Z}_p)$. In particular, we see

$$\begin{aligned}(\xi^{-1}t_p^\beta * v_{\lambda,j})[\left(\begin{smallmatrix} 1 & X \\ 0 & 1\end{smallmatrix}\right)] &= \langle\left(\begin{smallmatrix} 1 & \\ & w_n\end{smallmatrix}\right)\rangle_\lambda v_{\lambda,j}\left(\begin{pmatrix} 1 & -1+p^\beta X w_n \\ 0 & 1\end{pmatrix}\right) \qquad \text{(by defn. and (5.12))}\\ &= (-1)^{dnj}(\mathrm{N}_{F/\mathbf{Q}} \circ \det(-1+p^\beta X w_n))^j \left\langle\begin{pmatrix} -1+p^\beta X w_n & \\ & w_n\end{pmatrix}\right\rangle_\lambda \cdot v_\lambda^H \\ &\equiv \left\langle\left(\begin{smallmatrix} -1 & \\ & w_n\end{smallmatrix}\right)\right\rangle_\lambda \cdot v_\lambda^H \pmod{p^{\beta-\beta_0} V_\lambda^H(\mathcal{O}_L)} \qquad \text{(by (5.11)},\end{aligned}$$

which is visibly independent of j. Substituting this into (5.10), we obtain

$$(c_j - 1) \cdot \left[\left\langle\left(\begin{smallmatrix} -1 & \\ & w_n\end{smallmatrix}\right)\right\rangle_\lambda \cdot v_\lambda^H\right] \in p^{\beta-\beta_0} V_\lambda^H(\mathcal{O}_L). \tag{5.13}$$

As

$$\langle\left(\begin{smallmatrix} -1 & \\ & w_n\end{smallmatrix}\right)\rangle_\lambda \cdot v_\lambda^H \neq 0,$$

and this holds for all $\beta \geqslant \beta_0$, we deduce $c_j = 1$, completing the proof. □

In particular, all of our choices, and hence the maps $\mathcal{E}_\chi^{j,\eta_0}$, coincide with those in [38], so we may freely use the later results *ibid.* on the specific values of $\mathcal{E}_\chi^{j,\eta_0}$.

Remark 5.13. Proposition 5.12 would fail without the scalar $(-1)^{dnj}$ in Definition 5.10. If we had defined $\xi = \left(\begin{smallmatrix} 1 & -w_n \\ 0 & w_n \end{smallmatrix}\right)$ when defining $\mathrm{Ev}_{\beta,\delta}^M$, we would not need this scalar. However we choose $\xi = \left(\begin{smallmatrix} 1 & w_n \\ 0 & w_n \end{smallmatrix}\right)$, as chosen in [38], for compatibility with their results.

We now compare with the alignment of Jiang–Sun–Tian, who in [54] proved period relations at infinity for RASCARs. They fix a highest weight vector $v_\infty \in V_\lambda^\vee$, let $u = \left(\begin{smallmatrix} 1 & -w_n \\ w_n & 1 \end{smallmatrix}\right)$, and normalise the branching law† $\kappa_j^{\mathrm{JST}} : V_\lambda^\vee \to V_{j,-\mathsf{w}-j}^H$ so that $\kappa_j^{\mathrm{JST}}(u \cdot v_\infty) = 1$. Again, note all the κ_j^{JST} depend only on the choice of v_∞, which is well-defined up to scalar. Then we have:

Proposition 5.14. *We may choose v_∞ such that*

$$\kappa_{\lambda,j}^\circ = (\det w_n)^{jd} \cdot \kappa_j^{\mathrm{JST}}$$

for each $j \in \mathrm{Crit}(\lambda)$.

Proof. By Proposition 5.12 it suffices to show $\kappa_j^{\mathrm{JST}} = (\det w_n)^{jd} \cdot \kappa_j^{\mathrm{DJR}}$. As both lie in the same line, we know at least there exists $C_j \neq 0$ such that $\kappa_j^{\mathrm{DJR}} = C_j \kappa_j^{\mathrm{JST}}$. We want $C_j = (\det w_n)^{jd}$.

Note that

$$\xi = \left(\begin{smallmatrix} 1 & \\ & w_n \end{smallmatrix}\right) u \left(\begin{smallmatrix} 1 & -w_n \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} 1_n & \\ & -2^{-1}\cdot 1_n \end{pmatrix} w_{2n} \tag{5.14}$$

Note $w_{2n} \cdot v_0$ is a highest weight vector. Thus any $t \in T$ acts on $w_{2n} \cdot v_0$ as $\lambda^\vee(t)$, and any $n \in N$ acts trivially. Letting both sides of (5.14) act on v_0 thus gives

$$\xi \cdot v_0 = \lambda^\vee \left[\begin{pmatrix} 1_n & \\ & -2^{-1}\cdot 1_n \end{pmatrix}\right] \left(\begin{smallmatrix} 1 & \\ & w_n \end{smallmatrix}\right) u \cdot w_{2n} \cdot v_0 = (\det w_n)^{\mathsf{w}d} \left(\begin{smallmatrix} 1 & \\ & w_n \end{smallmatrix}\right) u \cdot v_\infty,$$

where we define $v_\infty := (\det w_n)^{-\mathsf{w}d} \lambda^\vee \left[\begin{pmatrix} 1_n & \\ & -2^{-1}\cdot 1_n \end{pmatrix}\right] \cdot w_{2n} \cdot v_0$. Then

$$\begin{aligned} 1 = \kappa_j^{\mathrm{DJR}}[\xi \cdot v_0] &= \det(w_n)^{\mathsf{w}d} \kappa_j^{\mathrm{DJR}}\left[\left(\begin{smallmatrix} 1 & \\ & w_n \end{smallmatrix}\right) u \cdot v_\infty\right] = (\det w_n)^{-jd} \kappa_j^{\mathrm{DJR}}[u \cdot v_\infty] \\ &= (\det w_n)^{-jd} C_j \kappa_j^{\mathrm{JST}}[u \cdot v_\infty] = (\det w_n)^{-jd} C_j. \end{aligned}$$

For this choice of v_∞ we have $C_j = (\det w_n)^{jd}$, as required. □

5.3. Non-vanishing of evaluation maps and Shalika models. We now show how classical evaluation maps can detect existence of Shalika models. Let π be any RACAR with attached maximal ideal $\mathfrak{m}_\pi \subset \mathcal{H}'$ as in §2.4. Let λ be the weight of π, with purity weight w.

Proposition 5.15. *Suppose there exists $\phi \in \mathrm{H}_c^t(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))_{\mathfrak{m}_\pi}^\epsilon$ such that*

$$\mathcal{E}_\chi^{j,\eta_0}(\phi) \neq 0 \tag{5.15}$$

for some χ, j and η_0. Then π admits a global (η,ψ)-Shalika model, where $\eta = \eta_0|\cdot|^{\mathsf{w}}$.

Proof. By Proposition 2.3, there exists a unique $\varphi_f \in \pi_f^K$ mapping to ϕ under (2.11). This isomorphism depended on a choice Ξ_∞^ϵ of generator of

$$\mathrm{H}^t(\mathfrak{g}_\infty, K_\infty^\circ; \pi_\infty \otimes V_\lambda^\vee(\mathbf{C}))^\epsilon \subset \left[\wedge^t(\mathfrak{g}_\infty/\mathfrak{t}_\infty)^\vee \otimes \pi_\infty \otimes V_\lambda^\vee(\mathbf{C})\right]^{K_\infty^\circ},$$

†To translate between this statement and ours here: observe that the torus defined in [54, (3.14)] is uTu^{-1}, where T is the usual torus; so the space they denote $(F_{\mathbb{K}}^\vee)^{\mathfrak{u}_{\mathbb{K}}}$ is $u\cdot(V_\lambda^\vee)^N$ here. But the space $(V_\lambda^\vee)^N$ of N-invariants is the highest weight space, so their v_0 is $u \cdot v_\infty$ here.

where $\mathfrak{t}_\infty = \mathrm{Lie}(T_\infty)$ and the inclusion is [26, II.3.4] (see also [45, §4.1]). Fixing bases $\{\omega_i\}$ of $(\mathfrak{g}_\infty/\mathfrak{t}_\infty)^\vee$ and $\{e_\alpha\}$ of $V_\lambda^\vee(\mathbf{C})$, there then exist unique vectors

$$\varphi^\epsilon_{\infty,\mathbf{i},\alpha} \in \pi_\infty \qquad \text{such that} \qquad \Xi^\epsilon_\infty = \sum_{\mathbf{i}}\sum_{\alpha} \omega_{\mathbf{i}} \otimes \varphi^\epsilon_{\infty,\mathbf{i},\alpha} \otimes e_\alpha,$$

where $\mathbf{i}$ ranges over tuples $(i_1, ..., i_t)$ and $\omega_{\mathbf{i}} = \omega_{i_1} \wedge \cdots \wedge \omega_{i_t}$. Define

$$\varphi^\epsilon_{\mathbf{i},\alpha} \coloneqq \varphi^\epsilon_{\infty,\mathbf{i},\alpha} \otimes \varphi_f.$$

By [38, Prop. 4.6], we see there exists an automorphic form

$$\varphi^\epsilon_{\phi,j} = \sum_{\mathbf{i}}\sum_{\alpha} a^\epsilon_{\mathbf{i},\alpha,j} \cdot \varphi^\epsilon_{\mathbf{i},\alpha} \quad \in \pi,$$

where the scalars $a^\epsilon_{\mathbf{i},\alpha,j} \in \mathbf{C}$ depend on $\kappa^\circ_{\lambda,j}$, and with $\mathbf{i}$ and α ranging over the same sets as above, such that

$$i_p^{-1}\big[\mathcal{E}^{j,\mathsf{w}}_{\beta,[\delta]}(\phi)\big] = \lambda(t_p^\beta) \int_{X_\beta[\delta]} \varphi^\epsilon_{\phi,j}(h\xi t_p^\beta)\ |\det(h_1^j h_2^{-\mathsf{w}-j})|_F\ dh.$$

Now arguing exactly as in the proof of [38, Thm. 4.7], we have an equality

$$i_p^{-1}\big[\mathcal{E}^{j,\eta_0}_\chi(\phi)\big] = \left[\gamma_{p\mathfrak{m}} \cdot \lambda(t_p^\beta) \prod_{\mathfrak{p}|p} \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{p})^{n^2\beta_{\mathfrak{p}}}\right] \cdot \Psi\left(j+\tfrac{1}{2},\ \varphi',\chi,\eta\right), \tag{5.16}$$

where

- $\gamma_{p\mathfrak{m}}$ is a non-zero volume constant defined in [38, (77)],
- $\varphi' \coloneqq (\xi t_p^\beta)\cdot \varphi^\epsilon_{\phi,j}$, and
- Ψ is the period integral defined in [41, Prop. 2.3].

Now, as in the proof of [41, Prop. 2.3], we may write

$$\begin{aligned}&\Psi\left(j+\tfrac{1}{2},\ \varphi',\chi,\eta\right) \qquad\qquad (5.17)\\ &= \int_{Z_n(\mathbf{Q})\backslash Z_n(\mathbf{A})}\left[\int_{H(\mathbf{Q})\backslash H^0} \varphi'\left[\left(\begin{smallmatrix} h_1 x & \\ & h_2\end{smallmatrix}\right)\right]\chi\left(\tfrac{\det h_1}{\det h_2}\right)\eta^{-1}(\det h_2)dh\right]\chi(x)|\det(x)|^j dx,\end{aligned}$$

where Z_n is the centre of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$ and

$$H^0 = \{(h_1,h_2) \in H(\mathbf{A}) : |\det(h_1)| = |\det(h_2)| = 1\}.$$

By (5.15), both (5.16) and (5.17) do not vanish; hence the inner integral of (5.17) also does not vanish. But existence of such a φ', χ and η implies π admits an (η,ψ)-Shalika model by [41, Prop. 2.2]. □

5.4. Local zeta integrals. In this and the next section, we state and prove Theorem 5.22, relating evaluation maps to L-values for our $\tilde{\pi}$. This is a compilation of results from [41, 45, 38, 54, 13]. First we relate to local zeta integrals in a general setting.

Let π be a RASCAR of $G(\mathbf{A})$, and $\chi = \prod \chi_v$ a Hecke character of F of conductor p^β. Recall $\Theta^{K,\epsilon}_{i_p} : \mathcal{S}^{\eta_f}_{\psi_f}(\pi_f^K) \to \mathrm{H}^t_c(S_K, \mathscr{V}_\lambda^\vee(\overline{\mathbf{Q}}_p))^\epsilon_{\mathfrak{m}_\pi}$ from (2.21), depending on a choice Ξ^ϵ_∞ at infinity. Attached to Ξ^ϵ_∞ and $j \in \mathrm{Crit}(\lambda)$ is a 'local zeta integral' $\zeta_{\infty,j}(\Xi^\epsilon_\infty)$, the quantity $\mathcal{P}_{\infty,j}(\Xi^\epsilon_\infty)$ from [54, (4.15)]. Recall the finite analogues $\zeta_v(-)$ from §2.6. Let

$$(\chi^j_{\mathrm{cyc}}\chi\eta)_\infty = [(-1)^j\chi_\sigma(-1)\eta_\sigma(-1)]_{\sigma\in\Sigma} \in \{\pm 1\}^\Sigma.$$

Lemma 5.16. *Let $W_f = \otimes_v W_v \in \mathcal{S}_{\psi_f}^{\eta_f}(\pi_f)$. If $\epsilon \neq (\chi_{\mathrm{cyc}}^j \chi\eta)_\infty$, then $\mathcal{E}_\chi^{j,\eta_0}(\Theta_{i_p}^{K,\epsilon}(W_f)) = 0$.*
If $\epsilon = (\chi_{\mathrm{cyc}}^j\chi\eta)_\infty$, then

$$i_p^{-1}\left(\mathcal{E}_\chi^{j,\eta_0}(\Theta_{i_p}^{K,\epsilon}(W_f))\right) = \left[\gamma_{p\mathfrak{m}} \cdot \lambda(t_p^\beta) \prod_{\mathfrak{p}|p} \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{p})^{n^2\beta_\mathfrak{p}}\right] \times \zeta_{\infty,j}(\Xi_\infty^\epsilon) \cdot \prod_{v\nmid p\infty} \zeta_v\Big(j+1/2; W_v, \chi_v\Big) \cdot \prod_{\mathfrak{p}|p} \zeta_\mathfrak{p}\Big(j+1/2; W_\mathfrak{p}\big(-\cdot \xi t_\mathfrak{p}^{\beta_\mathfrak{p}}\big), \chi_\mathfrak{p}\Big).$$

Proof. When $\epsilon \neq (\chi_{\mathrm{cyc}}^j\chi\eta)_\infty$, we deduce $\mathcal{E}_\chi^{j,\eta_0}(\phi_{\tilde{\pi}}^\epsilon) = 0$ as in the proof of [38, Thm. 4.7].

Suppose the sign condition is satisfied. We start from (5.16) above, where $\varphi' = \xi t_p^\beta \cdot \varphi_{\phi,j}^\epsilon$ in the notation *op. cit*, with $\phi = \Theta_{i_p}^K(W_f)$. Note $\mathcal{S}_\psi^\eta(\varphi') = W_{\infty,j}^\epsilon \otimes [\xi t_p^\beta \cdot W_f]$ for some $W_{\infty,j}^\epsilon \in \mathcal{S}_{\psi_\infty}^{\eta_\infty}(\pi_\infty)$. Now [38, §4.1.2] shows that $\Psi(j+1/2, \varphi', \chi, \eta)$ equals the product of local zeta integrals, as required. ☐

5.4.1. Local zeta integrals at infinity. At infinity, the following is a combination of Sun [77, Thm. 5.5], Jiang–Sun–Tian [54, Thm. 3.12], and Geng [42, Thm. 8.6].

Theorem 5.17. *Up to rescaling the basis elements $\Xi_\infty^\epsilon \in \mathrm{H}^t(\mathfrak{g}_\infty, K_\infty^\circ; \pi_\infty \otimes V_\lambda^\vee(\mathbf{C}))^\epsilon$, if $\epsilon = (\chi_{\mathrm{cyc}}^j\chi\eta)_\infty$, we have*

$$\zeta_{\infty,j}(\Xi_\infty^\epsilon) = i^{-jnd} \cdot L(\pi_\infty \otimes \chi_\infty, j+1/2).$$

Proof. For each $j \in \mathrm{Crit}(\lambda)$, Jiang–Sun–Tian construct a zeta integral $\zeta_{\infty,j}^{\mathrm{JST}}(\Xi_\infty^\epsilon)$ at infinity, and show it arises from an evaluation map/modular symbol process as above. Their main result is existence of $\varepsilon(\pi_\infty) = \prod_{\sigma\in\Sigma} \varepsilon(\pi_\sigma) \in \{\pm 1\}$ such that the quantity

$$\frac{\zeta_{\infty,j}^{\mathrm{JST}}(\Xi_\infty^\epsilon)}{i^{-jnd} \cdot L(\pi_\infty \times \chi_\infty, j+1/2) \cdot \varepsilon(\pi_\infty)^j} \tag{5.18}$$

is non-zero and independent of j when $\epsilon = (\chi_{\mathrm{cyc}}^j\chi\eta)_\infty$. Further, in [42], Geng shows that $\varepsilon(\pi_\sigma) = \det(w_n)$ for all σ, so $\varepsilon(\pi_\infty)^j = \det(w_n)^{jd}$.

The map $\zeta_{\infty,j}^{\mathrm{JST}}$ differs from $\zeta_{\infty,j}$ only in the choice of branching law, so by Proposition 5.14

$$\zeta_{\infty,j}(\Xi_\infty^\epsilon) = \det(w_n)^{jd} \cdot \zeta_{\infty,j}^{\mathrm{JST}}(\Xi_\infty^\epsilon). \tag{5.19}$$

Combining (5.18) and (5.19), we see

$$\frac{\zeta_{\infty,j}(\Xi_\infty^\epsilon)}{i^{-jnd} \cdot L(\pi_\infty \times \chi_\infty, j+1/2)} \tag{5.20}$$

is non-zero and independent of j. Now note $\zeta_{\infty,j}(\Xi_\infty^\epsilon)$ scales linearly with Ξ_∞^ϵ; so by rescaling the latter, we may assume (5.20) equals 1 for some j_0, hence for all j, as required. ☐

Definition 5.18. We let $e_\infty(\pi, \chi, j) := i^{-jnd} \cdot L(\pi_\infty \otimes \chi_\infty, j+1/2)$.

5.4.2. Local zeta integrals at p. Recall from §2.8 that we work in two local settings at p:

$(\mathrm{C2})_\mathfrak{p}$ $\pi_\mathfrak{p}$ is parahoric spherical admitting a Shalika model, $\tilde{\pi}_\mathfrak{p} = (\pi_\mathfrak{p}, \alpha_\mathfrak{p})$ is a Shalika Q-refinement, and $W_\mathfrak{p} \in \mathcal{S}_{\psi_\mathfrak{p}}^{\eta_\mathfrak{p}}(\pi_\mathfrak{p}^{J_\mathfrak{p}})[\![U_\mathfrak{p} - \alpha_\mathfrak{p}]\!]$ a generator.

$(\mathrm{C2}')_\mathfrak{p}$ $\pi_\mathfrak{p} = \mathrm{Ind}_B^G\theta_\mathfrak{p}$ is spherical, satisfies the hypotheses of Proposition 2.7, and $\tilde{\pi}_\mathfrak{p} = (\pi_\mathfrak{p}, \alpha_\mathfrak{p})$ is the Shalika Q-refinement from that result.

We will assume $(\mathrm{C2})_\mathfrak{p}$ throughout, and $(\mathrm{C2}')_\mathfrak{p}$ when considering unramified characters. For a quasi-character $\chi_\mathfrak{p}$ of $F_\mathfrak{p}^\times$, let $\tau(\chi_\mathfrak{p})$ be the local Gauss sum, normalised as in [13, §9.2].

Definition 5.19. Let $s \in \mathbf{C}$. If $\chi_{\mathfrak{p}}$ is ramified of conductor $\mathfrak{p}^{\beta_{\mathfrak{p}}}$, let $T(\chi_{\mathfrak{p}}) = \tau(\chi_{\mathfrak{p}})^n$ and

$$e'_{\mathfrak{p}}(\tilde{\pi}, \chi, s) := q_{\mathfrak{p}}^{\beta_{\mathfrak{p}} n\left(s+\frac{n}{2}-\frac{1}{2}\right)+\delta_{\mathfrak{p}} n\left(s-\frac{n}{2}-\frac{1}{2}\right)} \cdot \frac{q_{\mathfrak{p}}^n}{(q_{\mathfrak{p}}-1)^n}.$$

This depends only on $\beta_{\mathfrak{p}}$ and s, but we denote it this way for later consistency.

If $\chi_{\mathfrak{p}}$ is unramified and $(\mathrm{C2}')_{\mathfrak{p}}$ holds, let $T(\chi_{\mathfrak{p}}) = \chi(\varpi_{\mathfrak{p}})^{-n\delta_{\mathfrak{p}}}$ and

$$e'_{\mathfrak{p}}(\tilde{\pi}, \chi, s) := q_{\mathfrak{p}}^{\delta_{\mathfrak{p}} n\left(s-\frac{n}{2}-\frac{1}{2}\right)} \cdot \frac{q_{\mathfrak{p}}^n}{(q_{\mathfrak{p}}-1)^n} \cdot \alpha_{\mathfrak{p}} \cdot \prod_{i=n+1}^{2n} \frac{1-\theta_{\mathfrak{p},i}^{-1}\chi_{\mathfrak{p}}^{-1}(\varpi_{\mathfrak{p}})q_{\mathfrak{p}}^{s-\frac{1}{2}}}{1-\theta_{\mathfrak{p},i}\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})q_{\mathfrak{p}}^{-s-\frac{1}{2}}}.$$

Proposition 5.20. (D.–Januszewski–Raghuram; B.–D.–Graham–Jorza–W.).
Let $W_{\mathfrak{p}}$ be a generator of $\mathcal{S}_{\psi_{\mathfrak{p}}}^{\eta_{\mathfrak{p}}}(\pi_{\mathfrak{p}}^{J_{\mathfrak{p}}})[\![U_{\mathfrak{p}}-\alpha_{\mathfrak{p}}]\!]$.

(i) If $(C2)_{\mathfrak{p}}$ holds, then for all ramified quasi-characters $\chi_{\mathfrak{p}}$, we have

$$\zeta_{\mathfrak{p}}\big(s; W_{\mathfrak{p}}(-\cdot\xi t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}), \chi_{\mathfrak{p}}\big) = T(\chi_{\mathfrak{p}}) \cdot e'_{\mathfrak{p}}(\tilde{\pi}, \chi, s-\tfrac{1}{2}) \cdot W_{\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}). \tag{5.21}$$

(i) If $(C2')_{\mathfrak{p}}$ holds, (5.21) *also holds for unramified $\chi_{\mathfrak{p}}$.*

Proof. Given $(\mathrm{C2})_{\mathfrak{p}}$, (i) is [38, Prop. 3.4] (with a corrected power of $q_{\mathfrak{p}}$; see Appendix (2)).

If $(\mathrm{C2}')_{\mathfrak{p}}$ holds, (ii) was proved by the present authors with Graham and Jorza in [13, Prop. 9.3]. The only differences are that instead of $\xi = \left(\begin{smallmatrix} 1 & w_n \\ 0 & w_n \end{smallmatrix}\right)$ here, there is used $u^{-1} = \left(\begin{smallmatrix} 1 & -w_n \\ 0 & 1 \end{smallmatrix}\right)$; but we can compare the two integrals by noting that the integrand in [13] contains

$$\left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -w_n \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} t_p^{\beta} & \\ & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -hw_n & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & w_n \\ 0 & w_n \end{smallmatrix}\right)\left(\begin{smallmatrix} t_p^{\beta} & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -w_n & \\ & w_n \end{smallmatrix}\right).$$

The change of variables $h \leftrightarrow -hw_n$ removes the factor of $\chi_{\mathfrak{p}}(\det(-w_n))$ appearing in [13], and $\left(\begin{smallmatrix} -w_n & \\ & w_n \end{smallmatrix}\right)$ disappears by parahoric invariance. In [13] the term $W_{\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}})$ is denoted $F_0(w_{2n})$ and taken to be 1 (see §9.1 *ibid.*), so does not appear there. We also rearrange using $\alpha_{\mathfrak{p}} = q_{\mathfrak{p}}^{n^2/2}\theta_{\mathfrak{p},n+1}\cdots\theta_{\mathfrak{p},2n}(\varpi_{\mathfrak{p}})$. □

Remark 5.21. Proposition 5.20(i) holds assuming only $\tilde{\pi}_{\mathfrak{p}}$ is regular, rather than Shalika (i.e. without demanding that $W(t_{\mathfrak{p}}^{-\delta}) \neq 0$). In particular if $\tilde{\pi}_{\mathfrak{p}}$ is regular and $\zeta_{\mathfrak{p}}(s, W_{\mathfrak{p}}(-\cdot\xi t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}), \chi_{\mathfrak{p}}) \neq 0$ for some ramified $\chi_{\mathfrak{p}}$, then this result implies $\tilde{\pi}_{\mathfrak{p}}$ is Shalika.

5.5. Cohomological interpretation of L-values. Now suppose $\tilde{\pi}$ satisfies Conditions 2.8 or 2.8′, and recall

$$\phi_{\tilde{\pi}}^{\epsilon} = \Theta_{i_p}^{K(\tilde{\pi}),\epsilon}(W_f^{\mathrm{FJ}})/i_p(\Omega_{\pi}^{\epsilon})$$

from Definition 2.10. It is important that we now work at level $K = K(\tilde{\pi})$. The results from [41, 45, 38, 54, 13] combine to show:

Theorem 5.22. *Suppose $\tilde{\pi}$ satisfies Conditions 2.8′. Fix $\epsilon \in \{\pm 1\}^{\Sigma}$. Let χ be a finite order Hecke character of conductor $p^{\beta'}$, with $\beta' = (\beta'_{\mathfrak{p}})_{\mathfrak{p}|p}$ with each $\beta'_{\mathfrak{p}} \geqslant 0$. Let $\beta_{\mathfrak{p}} := \max(\beta'_{\mathfrak{p}}, 1)$. Let $j \in \mathrm{Crit}(\lambda)$. Then if $\epsilon \neq (\chi_{\mathrm{cyc}}^j\chi\eta)_{\infty}$, then $\mathcal{E}_{\chi}^{j,\eta_0}(\phi_{\tilde{\pi}}^{\epsilon}) = 0$. If $\epsilon = (\chi_{\mathrm{cyc}}^j\chi\eta)_{\infty}$, we have*

$$i_p^{-1}\left(\mathcal{E}_{\chi}^{j,\eta_0}(\phi_{\tilde{\pi}}^{\epsilon})\right) = \gamma_{p\mathfrak{m}} \cdot \lambda(t_p^{\beta}) \cdot \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d}^{(p)})^{jn} \cdot \tau(\chi_f)^n \times \left[\textstyle\prod_{\mathfrak{p}|p} e'_{\mathfrak{p}}(\tilde{\pi}, \chi, j)\right] \cdot e_{\infty}(\pi, \chi, j) \cdot \frac{L^{(p)}(\pi \otimes \chi, j+\frac{1}{2})}{\Omega_{\pi}^{\epsilon}}.$$

If $\tilde{\pi}$ satisfies the (more general) Conditions 2.8 then the same is true when $\beta'_{\mathfrak{p}} \geqslant 1$ for all $\mathfrak{p}|p$.

Here $\tau(\chi_f)$ is the Gauss sum, $\mathfrak{d}^{(p)}$ is the prime-to-p part of the different, and the $e(-)$ terms are as in Definitions 5.18 and 5.19 above.

Proof. By Lemma 5.16, we have vanishing unless the sign condition is satisfied, whence the left-hand side is a product of local zeta integrals. The integral at infinity was computed in Theorem 5.17. At $v \nmid p\infty$, the integral is $\zeta_v(j+1/2, W_v^{\mathrm{FJ}}, \chi_v)$, which was evaluated in (2.15). In the product, we get the claimed L-value and $\mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d}^{(p)})^{jn}$, and a product of $\chi_v(\varpi_v)$'s.

At $\mathfrak{p}|p$, if Conditions 2.8′ hold, then we are in case $(\mathrm{C2}')_{\mathfrak{p}}$ of Proposition 5.20, and this computes the integral for all $\chi_{\mathfrak{p}}$. If only Conditions 2.8 hold, then we are in case $(\mathrm{C2})_{\mathfrak{p}}$ and Proposition 5.20 computes it whenever $\beta_{\mathfrak{p}} \geqslant 1$.

The $T(\chi_{\mathfrak{p}})$'s combine with the products of $\chi_v(\varpi_v)$'s at $v \nmid p\infty$ to give $\tau(\chi_f)^n$, as in [38, Thm. 4.7]. The L-factors combine into $L^{(p)}(-)$. The other terms are as claimed. □

6. Finite slope p-adic L-functions

By a conjecture of Panchishkin [66], the p-adic L-function of $\tilde{\pi}$ is expected to be a p-adic distribution on Gal_p, the Galois group of the maximal abelian extension of F unramified outside $p\infty$, satisfying growth and interpolation properties. We now use the formalism of §4 to construct evaluation maps on overconvergent cohomology groups, valued in the space of distributions on Gal_p, and use them to give a construction of p-adic L-functions attached to non-Q-critical Q-refined RACARs $\tilde{\pi}$ satisfying Conditions 2.8. We show these p-adic L-functions satisfy the required growth and interpolation properties by using the results of §4.3 on the variation of evaluation maps. In particular, in this chapter we prove Theorem A of the introduction.

Since this chapter provides the technical heart of our p-adic interpolation results, for the convenience of the reader we briefly summarise its content.

- In §6.1, we set up the language of distributions on Gal_p, and endow them with an action of $H(\mathbf{A}_F)$, which will later allow us to use the formalism of §4.3 when combined with evaluation maps.
- In §6.2, we give the main technical construction and result of this chapter, namely the construction of the commutative diagram in Proposition 6.12. This yields a p-adic interpolation of the classical branching laws for $H \subset G$ described in §5.2.
- In §6.3, we define the overconvergent evaluation maps.
- In §6.4, we show that the overconvergent evaluations interpolate the classical evaluation maps $\mathcal{E}_\chi^{j,\eta_0}$ of the previous section.
- In §6.5, we recall growth properties on distributions on Gal_p, and prove that distributions in the image of our evaluation maps have controlled growth.
- In §6.6, we finally define the p-adic L-function of $\tilde{\pi}$ and prove Theorem A.

6.1. Distributions over Galois groups.

6.1.1. Definition of Galois distributions. Throughout this section, fix $\lambda_\pi \in X_0^*(T)$ a pure classical 'base' weight, and let
$$\Omega = \mathrm{Sp}(\mathcal{O}_\Omega) \subset \mathscr{W}_{\lambda_\pi}^Q$$
be an affinoid. We allow $\Omega = \{\lambda\}$ for λ classical, in which case $\mathcal{O}_\Omega = L$. Let $\chi_\Omega : T(\mathbf{Z}_p) \to \mathcal{O}_\Omega^\times$ be the tautological character attached to Ω, and recall the purity weight $\mathsf{w}_\Omega : \mathbf{Z}_p^\times \to \mathcal{O}_\Omega^\times$, all defined in §3.2.3.

We recall the structure of Gal_p. Recall $\mathcal{O}_{F,p} = \mathcal{O}_F \otimes \mathbf{Z}_p$. For $\beta = (\beta_\mathfrak{p})_{\mathfrak{p}|p}$ with $\beta_\mathfrak{p} > 0$ for each $\mathfrak{p} \mid p$, let

$$\mathscr{U}_\beta := [1 + p^\beta \mathcal{O}_{F,p}]/\overline{E(p^\beta)},$$

where $\overline{E(p^\beta)}$ is the p-adic closure of

$$E(p^\beta) := \{u \in \mathcal{O}_F^\times \cap F_\infty^{\times\circ} | u \equiv 1 \,(\mathrm{mod}\, p^\beta)\}.$$

Then by Class Field Theory we have an exact sequence

$$1 \to \mathscr{U}_\beta \xrightarrow{\iota} \mathrm{Gal}_p \xrightarrow{\jmath} \mathscr{C}\ell_F^+(p^\beta) \to 1. \tag{6.1}$$

Recall the distribution modules $\mathcal{D}(X, R)$ from §3.2. The sum of the natural restriction maps induces a decomposition

$$\mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega) \cong \bigoplus_{\mathbf{x} \in \mathscr{C}\ell_F^+(p^\beta)} \mathcal{D}(\mathrm{Gal}_p[\mathbf{x}], \mathcal{O}_\Omega), \tag{6.2}$$

where for $\mathbf{x} \in \mathscr{C}\ell_F^+(p^\beta)$, we define

$$\mathrm{Gal}_p[\mathbf{x}] := \jmath^{-1}(\mathbf{x}) \subset \mathrm{Gal}_p\,.$$

The map ι induces a map

$$\iota_* : \mathcal{D}(\mathscr{U}_\beta, \mathcal{O}_\Omega) \hookrightarrow \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega),$$

whose image can be identified with $\mathcal{D}(\mathrm{Gal}_p[\mathbf{1}_\beta], L)$, where $\mathbf{1}_\beta$ is the identity element in $\mathscr{C}\ell_F^+(p^\beta)$.

In the limit, the Artin reciprocity map $\mathrm{rec} : \mathbf{A}_F^\times \to \mathrm{Gal}_p$ induces an isomorphism

$$\mathrm{Gal}_p \xleftarrow[\sim]{\mathrm{rec}} \mathscr{C}\ell_F^+(p^\infty) := F^\times \backslash \mathbf{A}_F^\times / \overline{\mathscr{U}(p^\infty) F_\infty^{\times\circ}}, \tag{6.3}$$

where $\mathscr{U}(p^\infty) = \prod_{v \nmid p} \mathcal{O}_v^\times$. Note that the cyclotomic character χ_{cyc} from (5.1) is naturally a character on Gal_p; it is the character attached to the adelic norm via [16, §2.2.2].

6.1.2. Group actions on Galois distributions. If $c \in \mathbf{A}_F^\times$ and $x \in \mathrm{Gal}_p$, to simplify notation we write $cx := \mathrm{rec}(c)x$. We define a left action of $(\delta_1, \delta_2) \in \mathbf{A}_F^\times \times \mathbf{A}_F^\times$ on $\mathcal{A}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$ by

$$(\delta_1, \delta_2) * f(x) = \chi_{\mathrm{cyc}}(\delta_2)^{\mathsf{w}_\Omega} f(\delta_1^{-1}\delta_2 x), \tag{6.4}$$

and dually a left action on $\mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$. Recall the evaluation maps of §4 were indexed over $\pi_0(X_\beta)$ (a product of two class groups), and $\mathrm{pr}_\beta : \pi_0(X_\beta) \to \mathscr{C}\ell_F^+(p^\beta)$ from (5.3).

Lemma 6.1. *Let $\delta = (\delta_1, \delta_2) \in \mathbf{A}_F^\times \times \mathbf{A}_F^\times$, representing an element $[\delta] \in \pi_0(X_\beta)$, and let*

$$\mathbf{x} = \mathrm{pr}_\beta([\delta]) \in \mathscr{C}\ell_F^+(p^\beta).$$

The action of δ induces an isomorphism

$$\mathcal{D}(\mathscr{U}_\beta, \mathcal{O}_\Omega) \xrightarrow[\sim]{\iota_*} \mathcal{D}(\mathrm{Gal}_p[\mathbf{1}_\beta], \mathcal{O}_\Omega) \xrightarrow{\mu \,\mapsto\, \delta * \mu} \mathcal{D}(\mathrm{Gal}_p[\mathbf{x}], \mathcal{O}_\Omega).$$

Proof. The action of δ on $\mu \in \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$ is induced by the action of δ^{-1} on $\mathcal{A}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$ by

$$(\delta^{-1} * f)(x) = \chi_{\mathrm{cyc}}(\delta_2)^{-\mathsf{w}_\Omega} f(\delta_1 \delta_2^{-1} x).$$

By (4.4) $\delta_1\delta_2^{-1}$ is a representative of $\mathbf{x}$, so multiplication by $\delta_1\delta_2^{-1}$ on Gal_p sends $\mathrm{Gal}_p[\mathbf{1}_\beta]$ isomorphically to $\mathrm{Gal}_p[\mathbf{x}]$. Hence this action induces a map

$$\delta^{-1} * - : \mathcal{A}(\mathrm{Gal}_p[\mathbf{x}], \mathcal{O}_\Omega) \to \mathcal{A}(\mathrm{Gal}_p[\mathbf{1}_\beta], \mathcal{O}_\Omega)$$

which dualises to the claimed map. □

Via (6.4), we have an action of $H(\mathbf{A})$ on $f \in \mathcal{A}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$ by

$$(h_1, h_2) * f := (\det(h_1), \det(h_2)) * f, \tag{6.5}$$

and hence a dual action on $\mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$. Note that both $H(\mathbf{Q})$ and H_∞° act trivially.

6.2. p-adic interpolation of branching laws for $H \subset G$. Let $K \subset G(\mathbf{A}_f)$ be an open compact subgroup as in §4.2. For each β with $\beta_{\mathfrak{p}} \geqslant 1$ for all $\mathfrak{p}|p$, in §6.3 we will define a map

$$\mathrm{Ev}_\beta^{\eta_0} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega) \to \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$$

that simultaneously interpolates the evaluation maps $\mathcal{E}_\chi^{j,\eta_0}$ of §5 for all classical $\lambda \in \Omega$, for all $j \in \mathrm{Crit}(\lambda)$, and for all χ of conductor p^β.

The key step in the construction of this map, which we pursue in this subsection, is to interpolate the branching law of Lemma 5.2 (which was crucially used in the definition of $\mathcal{E}_\chi^{j,\eta_0}$). We do this by interpolating the maps $\kappa_{\lambda,j}^\circ$ from (5.2), in the following sense: for classical $\lambda \in \Omega$, by Lemma 4.6 we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega) & \xrightarrow{\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\Omega}} & (\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}} & & \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega) \\
\downarrow{\scriptstyle r_\lambda \circ \mathrm{sp}_\lambda} & & \downarrow{\scriptstyle r_\lambda \circ \mathrm{sp}_\lambda} & & \downarrow{\scriptstyle \mu \mapsto \mathrm{sp}_\lambda(\mu)(\chi_{\mathrm{cyc}}^j)} \\
\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_\lambda^\vee) & \xrightarrow{\mathrm{Ev}_{\beta,\delta}^{V_\lambda^\vee}} & (V_\lambda^\vee)_{\Gamma_{\beta,\delta}} & \xrightarrow{\kappa_{\lambda,j}^\circ} & L,
\end{array} \tag{6.6}$$

and we now define the 'missing' horizontal map in the top row so that the horizontal compositions commute with the outer vertical maps (see Proposition 6.12).

Our strategy is to construct a map

$$v_\Omega^\beta : \mathcal{A}(\mathrm{Gal}_p, \mathcal{O}_\Omega) \to \mathcal{A}_\Omega^{\Gamma_{\beta,\delta}} \subset \mathcal{A}_\Omega, \tag{6.7}$$

where $\mathcal{A}_\Omega^{\Gamma_{\beta,\delta}}$ denotes the $\Gamma_{\beta,\delta}$-invariants, and then dualise to get the required map $(\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}} \to \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$.

The map $\kappa_{\lambda,j}^\circ$ was defined by an element $v_{\lambda,j} \in V_\lambda$, which we described explicitly in Lemma 5.7 (noting $v_{\lambda,j}$ was defined to be $\kappa_{\lambda,j}^\vee(u_j^\vee)$). Our definition of v_Ω^β, given in (6.10), is really an interpolation of this last description of $v_{\lambda,j}$.

Remark. Note restricting elements of $\mathcal{D}_\Omega$ to the subspace $\mathcal{A}_\Omega^{\Gamma_{\beta,\delta}} \subset \mathcal{A}_\Omega$ induces a well-defined map $(\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}} \to (\mathcal{A}_\Omega^{\Gamma_{\beta,\delta}})^\vee$. Slightly abusing notation/terminology, we will identify elements of $(\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}}$ with their image under this map, and continue to call them distributions (on $\mathcal{A}^{\Gamma_{\beta,\delta}}$).

6.2.1. Support conditions on distributions. We want to define v_Ω^β to interpolate $v_{\lambda,j} \in V_\lambda$ from §5.2. However, we have explicitly described the function $v_{\lambda,j} : G(\mathbf{Z}_p) \to L$ only on the subset $N_Q^\times(\mathbf{Z}_p) \subset G(\mathbf{Z}_p)$ (see Lemma 5.7). The following support condition shows that for the outer vertical maps of (6.6) to commute with the horizontal compositions, it is sufficient to specify v_Ω^β on subsets $N_Q^\beta(\mathbf{Z}_p)$ of $N_Q^\times(\mathbf{Z}_p)$.

For $\beta = (\beta_{\mathfrak{p}})_{\mathfrak{p}|p}$ with each $\beta_{\mathfrak{p}} \geqslant 1$, let

$$N_Q^\beta(\mathbf{Z}_p) := \left\{ \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in N_Q(\mathbf{Z}_p) : X \equiv -I_n \pmod{p^\beta} \right\} \subset N_Q^\times(\mathbf{Z}_p), \tag{6.8}$$

and define

$$J_p^\beta := (N_Q^-(\mathbf{Z}_p) \cap J_p) \cdot H(\mathbf{Z}_p) \cdot N_Q^\beta(\mathbf{Z}_p) \subset J_p.$$

Lemma 6.2. *Let $\Phi \in \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)$, and let $\delta \in H(\mathbf{A})$. The distribution $\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\Omega}(\Phi) \in (\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}}$ has support in J_p^β, in the sense that if $f \in \mathcal{A}_\Omega^{\Gamma_{\beta,\delta}}$, then*

$$\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\Omega}(\Phi)(f) = \mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\Omega}(\Phi)\left(f|_{J_p^\beta}\right)$$

depends only on the restriction of f to J_p^β.

Proof. Via the map τ_β°, we see that

$$\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\Omega}(\Phi) \in (\xi t_p^\beta * \mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}}.$$

It thus suffices to prove that for any $\mu \in \mathcal{D}_\Omega$ and $f \in \mathcal{A}_\Omega$, we have

$$(\xi t_p^\beta * \mu)(f) = (\xi t_p^\beta * \mu)(f|_{J_p^\beta}),$$

or equivalently that

$$(\xi t_p^\beta)^{-1} * f = (\xi t_p^\beta)^{-1} * f|_{J_p^\beta}.$$

By definition (see §3.3), the action of $(\xi t_p^\beta)^{-1}$ on $f \in \mathcal{A}_\Omega$ is induced by the action

$$\begin{aligned} \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) &\longmapsto \left[t_p^\beta \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) t_p^{-\beta}\right] \xi^{-1} \\ &= \begin{pmatrix} 1 & p^\beta X \\ 0 & 1 \end{pmatrix} \left(\begin{smallmatrix} I_n & -I_n \\ 0 & w_n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & w_n \end{smallmatrix}\right) \begin{pmatrix} 1 & -I_n + p^\beta X w_n \\ 0 & 1 \end{pmatrix} \in J_p^\beta \end{aligned} \tag{6.9}$$

on $\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in N_Q(\mathbf{Z}_p)$. Thus $((\xi t_p^\beta)^{-1} * f)|_{N_Q(\mathbf{Z}_p)}$ depends only on $f|_{J_p^\beta}$. By parahoric decomposition (3.8), we deduce that $(\xi t_p^\beta)^{-1} * f$ depends only on $f|_{J_p^\beta}$, as claimed. □

6.2.2. Interpolation of v_λ^H in families. Recall that the description of $v_{\lambda,j}$ in Lemma 5.7 was given in terms of a specific vector $v_\lambda^H \in V_\lambda^H$. We now interpolate $v_{\lambda,j}$ as λ varies in Ω.

In Notation 5.9, we fixed $v_{\lambda_\pi}^H \in V_{\lambda_\pi}^H(\mathcal{O}_L)$ to be an (optimally integral) generator of the unique line in $V_{\lambda_\pi}^H(L)$ on which the action of $\langle \left(\begin{smallmatrix} h & \\ & h \end{smallmatrix}\right) \rangle_{\lambda_\pi}$ is multiplication by $(\mathrm{N}_{F/\mathbf{Q}} \circ \det)^{\mathsf{w}_{\lambda_\pi}}$.

Notation 6.3. Let $v_\Omega^H := v_{\lambda_\pi}^H \otimes 1 \in V_\Omega^H$.

The following statement is an analogue of Lemma 5.8 for families.

Lemma 6.4. *Let $h \in G_n(\mathbf{Z}_p)$. Then*

$$\left\langle \left(\begin{smallmatrix} h & \\ & h \end{smallmatrix}\right) \right\rangle_\Omega \cdot v_\Omega^H = \mathsf{w}_\Omega(\mathrm{N}_{F/\mathbf{Q}} \circ \det(h))\; v_\Omega^H.$$

Proof. By the definition of the action of $H(\mathbf{Z}_p)$ on V_Ω^H (see (3.5)), we have

$$\left\langle \left(\begin{smallmatrix} h & \\ & h \end{smallmatrix}\right) \right\rangle_\Omega \cdot (v_{\lambda_\pi}^H \otimes 1) = \mathsf{w}_{\lambda_\pi}(\mathrm{N}_{F/\mathbf{Q}} \circ \det(h)) v_{\lambda_\pi}^H \otimes \mathsf{w}_{\Omega_0}(\mathrm{N}_{F/\mathbf{Q}} \circ \det(h)),$$

recalling

$$\chi_{\Omega_0}(h,h) = \mathsf{w}_{\Omega_0}(\mathrm{N}_{F/\mathbf{Q}} \circ \det(h))$$

from (3.4). We conclude as $\mathsf{w}_\Omega = \mathsf{w}_{\lambda_\pi}\mathsf{w}_{\Omega_0}$. □

Lemma 6.5. *If $\lambda \in \Omega$ is a classical weight, then $\mathrm{sp}_\lambda(v_\Omega^H) \in V_\lambda^H(\mathcal{O}_L)$ is optimally integral, non-zero, and*

$$\left\langle \left(\begin{smallmatrix} h & \\ & h \end{smallmatrix}\right) \right\rangle_\lambda \cdot \mathrm{sp}_\lambda(v_\Omega^H) = (\mathrm{N}_{F/\mathbf{Q}} \circ \det(h))^{\mathsf{w}_\lambda} \mathrm{sp}_\lambda(v_\Omega^H).$$

Proof. Non-vanishing is immediate from the definition, and the action property follows from specialising Lemma 6.4. To see $\mathrm{sp}_\lambda(v_\Omega^H)$ is integral, recall $v_{\lambda_\pi}^H \in V_{\lambda_\pi}^H(\mathcal{O}_L)$ is integral, so $v_{\lambda_\pi}^H(H(\mathbf{Z}_p)) \subset \mathcal{O}_L$. Since λ is algebraic, we have $\lambda\lambda_\pi^{-1}(H(\mathbf{Z}_p)) \subset \mathcal{O}_L^\times$. By definition $\mathrm{sp}_\lambda(v_\Omega^H) = v_{\lambda_\pi}^H \otimes \lambda\lambda_\pi^{-1}$, so we deduce

$$\mathrm{sp}_\lambda(v_\Omega^H)(H(\mathbf{Z}_p)) \subset \mathcal{O}_L.$$

As $v_{\lambda_\pi}^H$ is optimally integral, it also follows that

$$\varpi_L^{-1}\mathrm{sp}_\lambda(v_\Omega^H)(H(\mathbf{Z}_p)) \not\subset \mathcal{O}_L,$$

so $\mathrm{sp}_\lambda(v_\Omega^H)$ is optimally integral as claimed. □

Lemma 6.5 allows us to make the following renormalisation of the vectors from Notation 5.9, which aligns them in the family Ω.

Definition 6.6. If $\lambda \in \Omega$ is a classical weight, let

$$v_\lambda^H := \mathrm{sp}_\lambda(v_\Omega^H) \in V_\lambda^H(\mathcal{O}_L).$$

Remark 6.7. This does not change our earlier choice of $v_{\lambda_\pi}^H$, since $\mathrm{sp}_{\lambda_\pi}^H(v_\Omega^H) = v_{\lambda_\pi}^H$.

From v_λ^H, as in §5.2 we obtain compatible choices of $\kappa_{\lambda,j}^\circ$ as j varies in $\mathrm{Crit}(\lambda)$. The definition of v_λ^H depends only on v_Ω^H, which depends only on the choice of $v_{\lambda_\pi}^H$. In particular, the (single) choice of $v_{\lambda_\pi}^H$ determines compatible choices of $\kappa_{\lambda,j}^\circ$ for all classical $\lambda \in \Omega$ and all $j \in \mathrm{Crit}(\lambda)$.

6.2.3. Construction of v_Ω^β and κ_Ω^β. Note

$$\mathcal{A}(\mathscr{U}_\beta, \mathcal{O}_\Omega) \subset \mathcal{A}(1 + p^\beta \mathcal{O}_{F,p}, \mathcal{O}_\Omega)$$

is the subset of functions invariant under $\overline{E(p^\beta)}$. Recall $N_Q^\beta(\mathbf{Z}_p)$ from (6.8). We have a map

$$\begin{aligned} N_Q^\beta(\mathbf{Z}_p) &\longrightarrow 1 + p^\beta \mathcal{O}_{F,p} \\ \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) &\longmapsto (-1)^n \det(X). \end{aligned}$$

Define a map

$$v_\Omega^\beta : \mathcal{A}(1 + p^\beta \mathcal{O}_{F,p}, \mathcal{O}_\Omega) \longrightarrow \mathcal{A}_\Omega$$

as follows. For $f \in \mathcal{A}(1 + p^\beta \mathcal{O}_{F,p}, \mathcal{O}_\Omega)$, define

$$v_\Omega^\beta(f) : N_Q(\mathbf{Z}_p) \to V_\Omega^H$$

by setting, for $X \in \mathrm{M}_n(\mathcal{O}_{F,p})$,

$$v_\Omega^\beta(f)\left(\begin{smallmatrix} I_n & X \\ 0 & I_n \end{smallmatrix}\right) = \begin{cases} f\big((-1)^n \det(X)\big)\Big(\big\langle\left(\begin{smallmatrix} X & 0 \\ 0 & I_n \end{smallmatrix}\right)\big\rangle_\Omega \cdot v_\Omega^H\Big) & : \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in N_Q^\beta(\mathbf{Z}_p), \\ 0 & : \text{else}. \end{cases} \tag{6.10}$$

Extending under the parahoric decomposition using (3.8) determines $v_\Omega^\beta(f)$ as an element of $\mathcal{A}_\Omega$.

Definition 6.8. Dualising

$$\begin{aligned} \mathcal{A}(\mathscr{U}_\beta, \mathcal{O}_\Omega) \subset \mathcal{A}(1 + p^\beta \mathcal{O}_{F,p}, \mathcal{O}_\Omega) &\longrightarrow \mathcal{A}_\Omega, \\ f &\longmapsto v_\Omega^\beta(f) \end{aligned}$$

gives a map

$$\kappa_\Omega^\beta : \mathcal{D}_\Omega \longrightarrow \mathcal{D}(\mathscr{U}_\beta, \mathcal{O}_\Omega). \tag{6.11}$$

Remark 6.9. The map κ_Ω^β, combined with Lemma 6.1, will induce the 'missing' map in (6.6). To motivate (6.10) and Definition 6.8, compare to the description of $\kappa_{\lambda,j}^\circ$ in Lemma 5.11. For the support condition in (6.10), note that for the outer maps of (6.6) to commute, by Lemma 6.2 it suffices to consider $v_\Omega^\beta(f)$ supported on J_p^β, and hence (by parahoric decomposition) on $N_Q^\beta(\mathbf{Z}_p)$.

Restricting under (6.1), we may see χ_{cyc} as an element of $\mathcal{A}(\mathscr{U}_\beta, L)$, and thus make sense of $v_\lambda^\beta(\chi_{\mathrm{cyc}}^j)$, supported on J_p^β. Let $\lambda \in \Omega$ be classical. Recall

$$v_{\lambda,j} : N_Q^\times(\mathbf{Z}_p) \to V_\lambda^H(L)$$

from Lemma 5.11. In (5.8) of this lemma, we normalise v_λ^H as in Definition 6.6. The following shows that v_λ^β interpolates $v_{\lambda,j}$ as j varies in $\mathrm{Crit}(\lambda)$, and hence interpolates branching laws in the 'cyclotomic direction'.

Lemma 6.10. *Let $\lambda \in \Omega$ be classical. For all $j \in \mathrm{Crit}(\lambda)$, we have*

$$v_\lambda^\beta(\chi_{\mathrm{cyc}}^j)\big|_{J_p^\beta} = v_{\lambda,j}\big|_{J_p^\beta}.$$

Proof. If $g = n_Q^- h \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in J_p^\beta$, then

$$\det(X) \in (-1)^n + p^\beta \mathcal{O}_{F,p} \subset (\mathcal{O}_{F,p})^\times,$$

where the last inclusion follows as $\beta_{\mathfrak{p}} \geqslant 1$ for all $\mathfrak{p}$. Hence

$$\mathrm{N}_{F/\mathbf{Q}} \circ \det(X) \in \mathbf{Z}_p^\times, \qquad \text{so} \qquad \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in N_Q^\times(\mathbf{Z}_p).$$

On such X we have

$$\chi_{\mathrm{cyc}}^j((-1)^n \det(X)) = (-1)^{dnj} \mathrm{N}_{F/\mathbf{Q}} \circ \det(X)^j.$$

Combining this and the definition of v_λ^β with Lemma 5.11, we see that

$$v_\lambda^\beta(\chi_{\mathrm{cyc}}^j)\left[\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)\right] = v_{\lambda,j}\left[\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)\right]$$

for all $\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) \in N_Q^\beta(\mathbf{Z}_p)$. We conclude $v_\lambda^\beta(\chi_{\mathrm{cyc}}^j)$ and $v_{\lambda,j}$ agree on all of J_p^β, as they satisfy the same transformation law under parahoric decomposition. ☐

We now combine v_Ω^β with the formalism of evaluation maps developed in §4.3.

Proposition 6.11. *(i) The action of $\ell = (\ell_1, \ell_2) \in L_\beta$ on $\mathcal{A}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$ under (6.5) is by*

$$\begin{aligned} [\ell * f](x) &:= [(\det(\ell_1), \det(\ell_2)) * f](x) \\ &= \mathrm{N}_{F/\mathbf{Q}}(\det(\ell_{2,p}))^{\mathsf{w}_\Omega} f(\det(\ell_{1,p}^{-1}\ell_{2,p})x). \end{aligned} \tag{6.12}$$

It preserves $\mathcal{A}(\mathscr{U}_\beta, \mathcal{O}_\Omega)$, giving it the structure of an L_β-module.

(ii) The map

$$\begin{aligned} \mathcal{A}(\mathscr{U}_\beta, \mathcal{O}_\Omega) &\longrightarrow \mathcal{A}_\Omega, \\ f &\longmapsto v_\Omega^\beta(f) \end{aligned}$$

is a map of L_β-modules.

(iii) The image of v_Ω^β is a subspace of the $\Gamma_{\beta,\delta}$-invariants $\mathcal{A}_\Omega^{\Gamma_{\beta,\delta}}$.

(iv) The map κ_Ω^β from Definition 6.8 is a map of left L_β-modules, and factors through

$$\kappa_\Omega^\beta : (\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}} \to \mathcal{D}(\mathscr{U}_\beta, \mathcal{O}_\Omega).$$

Proof. (i) Since $\det(\ell_i) \in (\mathcal{O}_F \otimes \widehat{\mathbf{Z}})^\times$, we have

$$\chi_{\mathrm{cyc}}(\det(\ell_2)) = \mathrm{N}_{F/\mathbf{Q}}(\det(\ell_{2,p}))$$

and $\det(\ell_{i,v}) \in \mathscr{U}(p^\infty)$ for all $v \nmid p\infty$. Hence

$$[\det(\ell_1^{-1}\ell_2)x] = [\det(\ell_{1,p}^{-1}\ell_{2,p}^{-1})x]$$

in Gal_p, and (6.4) induces the stated action. It preserves $\mathcal{A}(\mathscr{U}_\beta, \mathcal{O}_\Omega)$ since $\det(\ell_{1,p}^{-1}\ell_{2,p}) \equiv 1 \pmod{p^\beta}$ by [38, Lem. 2.1].

(ii) For $f \in \mathcal{A}(\mathscr{U}_\beta, L)$, we must show that

$$\ell * v_\Omega^\beta(f) = v_\Omega^\beta(\ell * f).$$

Let $X \in \mathrm{M}_n(\mathcal{O}_{F,p})$. If $\det(X) \not\equiv (-1)^n \pmod{p^\beta}$, both sides are zero at $\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)$. If $\det(X) \equiv (-1)^n \pmod{p^\beta}$, then

$$\begin{aligned}
(\ell * v_\Omega^\beta(f))\left(\begin{smallmatrix} I_n & X \\ 0 & I_n \end{smallmatrix}\right) &= v_\Omega^\beta(f)\left(\begin{smallmatrix} \ell_{1,p} & X\ell_{2,p} \\ 0 & \ell_{2,p} \end{smallmatrix}\right) \\
&= \left\langle\left(\begin{smallmatrix} \ell_{1,p} & 0 \\ 0 & \ell_{2,p} \end{smallmatrix}\right)\right\rangle_\Omega \cdot v_\Omega^\beta(f)\left(\begin{smallmatrix} I_n & \ell_{1,p}^{-1}X\ell_{2,p} \\ 0 & I_n \end{smallmatrix}\right) \\
&= \left\langle\left(\begin{smallmatrix} \ell_{1,p} & 0 \\ 0 & \ell_{2,p} \end{smallmatrix}\right)\right\rangle_\Omega \cdot \left\langle\left(\begin{smallmatrix} \ell_{1,p}^{-1}X\ell_{2,p} & 0 \\ 0 & I_n \end{smallmatrix}\right)\right\rangle_\Omega \cdot \left(f\left[(-1)^n \det(\ell_{1,p}^{-1}X\ell_{2,p})\right]v_\Omega^H\right) \\
&= \left\langle\left(\begin{smallmatrix} X & 0 \\ 0 & I_n \end{smallmatrix}\right)\right\rangle_\Omega \cdot \left\langle\left(\begin{smallmatrix} \ell_{2,p} & 0 \\ 0 & \ell_{2,p} \end{smallmatrix}\right)\right\rangle_\Omega \cdot \left(f\left[(-1)^n \det(\ell_{1,p}^{-1}X\ell_{2,p})\right]v_\Omega^H\right) \\
&= \left\langle\left(\begin{smallmatrix} X & 0 \\ 0 & I_n \end{smallmatrix}\right)\right\rangle_\Omega \cdot \left(\mathrm{N}_{F/\mathbf{Q}}(\det(\ell_{2,p}))^{\mathsf{w}_\Omega} f\left[(-1)^n \det(\ell_{1,p}^{-1}\ell_{2,p})\det(X)\right]v_\Omega^H\right) \\
&= \left\langle\left(\begin{smallmatrix} X & 0 \\ 0 & I_n \end{smallmatrix}\right)\right\rangle_\Omega \cdot \left((\ell * f)\left[(-1)^n \det(X)\right]v_\Omega^H\right) \\
&= v_\Omega^\beta(\ell * f)\left(\begin{smallmatrix} I_n & X \\ 0 & I_n \end{smallmatrix}\right),
\end{aligned}$$

proving (ii); the first equality is the $*$-action, the second is (3.8), the third is (6.10), the fifth is Lemma 6.4, the sixth by (i), and the last is (6.10).

(iii) Note $\Gamma_{\beta,\delta} \subset H(\mathbf{Q})$ acts trivially on $\mathcal{A}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$ (see (6.5)). Hence $\delta^{-1}\Gamma_{\beta,\delta}\delta \subset L_\beta$ acts trivially on $\mathcal{A}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$, hence trivially on $\mathcal{A}(\mathscr{U}_\beta, \mathcal{O}_\Omega)$. From (ii), it follows that $\delta^{-1}\Gamma_{\beta,\delta}\delta$ – acting as a subgroup of L_β – acts trivially on the image of v_Ω^β. But by definition of the $\Gamma_{\beta,\delta}$-action (see (4.7)), this means $\Gamma_{\beta,\delta}$ acts trivially on this image.

(iv) That κ_Ω^β is a map of L_β-modules follows from (ii), and thus it factors through $(\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}}$ since the target is $\Gamma_{\beta,\delta}$-invariant by (iii). □

6.2.4. Proof that κ_Ω^β interpolates $\kappa_{\lambda,j}^\circ$. The following is the main result of §6.2.

Proposition 6.12. *Let $\lambda \in \Omega$ classical and $j \in \mathrm{Crit}(\lambda)$. The following diagram commutes:*

$$\begin{array}{ccccc}
\mathrm{H}_c^t(S_K, \mathscr{D}_\Omega) & \xrightarrow{\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\Omega}} & (\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}} & \xrightarrow{\kappa_\Omega^\beta} & \mathcal{D}(\mathscr{U}_\beta, \mathcal{O}_\Omega) \\
\downarrow \mathrm{sp}_\lambda & & \downarrow \mathrm{sp}_\lambda & & \downarrow \mathrm{sp}_\lambda \\
\mathrm{H}_c^t(S_K, \mathscr{D}_\lambda) & \xrightarrow{\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\lambda}} & (\mathcal{D}_\lambda)_{\Gamma_{\beta,\delta}} & \xrightarrow{\kappa_\lambda^\beta} & \mathcal{D}(\mathscr{U}_\beta, L) \\
\downarrow r_\lambda & & & & \downarrow \int_{\mathscr{U}_\beta} \chi_{\mathrm{cyc}}^j \\
\mathrm{H}_c^t(S_K, \mathscr{V}_\lambda^\vee) & \xrightarrow{\mathrm{Ev}_{\beta,\delta}^{V_\lambda^\vee}} & (V_\lambda^\vee)_{\Gamma_{\beta,\delta}} & \xrightarrow{\kappa_{\lambda,j}^\circ} & L.
\end{array} \tag{6.13}$$

Proof. The top left-hand square commutes by Lemma 4.6. We next consider the top-right square. In the definition of κ_Ω^β, note by definition that $\mathrm{sp}_\lambda(v_\Omega^H) = v_\lambda^H$ and the action $\langle\cdot\rangle_\Omega$ specialises to $\langle\cdot\rangle_\lambda$ under sp_λ. In particular, if $f_\lambda \in \mathcal{A}(\mathscr{U}_\beta, L)$ and $f_\Omega \in \mathcal{A}(\mathscr{U}_\beta, \mathcal{O}_\Omega)$ is any lift under sp_λ, then

$$\mathrm{sp}_\lambda[v_\Omega^\beta(f_\Omega)] = v_\lambda^\beta(f_\lambda) \in \mathcal{A}_\lambda.$$

We describe the map $\mathrm{sp}_\lambda : \mathcal{D}_\Omega \to \mathcal{D}_\lambda$ directly. Let $\mu_\Omega \in \mathcal{D}_\Omega$, and $g_\lambda \in \mathcal{A}_\lambda$. Choose any $g_\Omega \in \mathcal{A}_\Omega$ with $\mathrm{sp}_\lambda(g_\Omega) = g_\lambda$. Then

$$[\mathrm{sp}_\lambda(\mu_\Omega)](g_\lambda) = \mathrm{sp}_\lambda[\mu_\Omega(g_\Omega)] \in \mathcal{O}_\Omega/\mathfrak{m}_\lambda,$$

which is easily seen to be independent of lift. Here $\mathfrak{m}_\lambda \subset \mathcal{O}_\Omega$ is the maximal ideal attached to λ. In particular, for f_λ, f_Ω as above,

$$\mathrm{sp}_\lambda(\mu_\Omega)[v_\lambda^\beta(f_\lambda)] = \mathrm{sp}_\lambda[\mu_\Omega(v_\Omega^\beta(f_\Omega))].$$

Let $\mu_\Omega \in (\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}}$; then the top right square commutes as

$$\begin{aligned}[\kappa_\lambda^\beta \circ \mathrm{sp}_\lambda(\mu_\Omega)](f_\lambda) = \mathrm{sp}_\lambda(\mu_\Omega)\big[v_\lambda^\beta(f_\lambda)\big] &= \mathrm{sp}_\lambda\big[\mu_\Omega\big(v_\Omega^\beta(f_\Omega)\big)\big]\\ &= \mathrm{sp}_\lambda\big[(\kappa_\Omega^\beta(\mu_\Omega))(f_\Omega)\big] = \big[\mathrm{sp}_\lambda \circ \kappa_\Omega^\beta(\mu_\Omega)\big](f_\lambda).\end{aligned}$$

We have used the previous paragraph in the second equality.

Now consider the bottom rectangle. If we 'complete' (6.13) with the natural map $r_\lambda : (\mathcal{D}_\lambda)_{\Gamma_{\beta,\delta}} \to (V_\lambda^\vee)_{\Gamma_{\beta,\delta}}$, then the bottom-left square would commute by Lemma 4.6, but the bottom-right square would *not* commute, due to support conditions. However if $\mu \in \mathrm{Im}(\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\lambda})$, then μ is supported on J_p^β by Lemma 6.2. For such μ, compute

$$\begin{aligned}\int_{\mathscr{U}_\beta} \chi_{\mathrm{cyc}}^j \cdot \kappa_\lambda^\beta(\mu) &= \int_{J_p} v_\lambda^\beta(\chi_{\mathrm{cyc}}^j)\cdot\mu\\ &= \int_{J_p^\beta} v_\lambda^\beta(\chi_{\mathrm{cyc}}^j)\cdot\mu = \int_{J_p^\beta} v_{\lambda,j}\cdot\mu = \int_{G(\mathbf{Z}_p)} v_{\lambda,j}\cdot r_\lambda(\mu)\end{aligned}$$

(recalling from Lemma 5.11 that $\kappa_{\lambda,j}^\circ$ is evaluation at $v_{\lambda,j}$). In the second equality, we use that μ has support on J_p^β, whence the third equality follows from Lemma 6.10. In the last, because $v_{\lambda,j} \in V_\lambda$ we have $\mu(v_{\lambda,j}) = r_\lambda(\mu)(v_{\lambda,j})$, and then we expand from J_p^β to $G(\mathbf{Z}_p)$ using that μ (hence $r_\lambda(\mu)$) is supported on J_p^β again. Thus the bottom-right square is commutative on the image of $\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\lambda}$, and the bottom rectangle is commutative. □

6.3. Distribution-valued evaluation maps. We now define overconvergent analogues of $\mathcal{E}_\chi^{j,\eta_0}$. Let

- $\delta = (\delta_1, \delta_2) \in H(\mathbf{A})$,
- let $[\delta]$ be its class in $\pi_0(X_\beta)$, and
- let $\mathbf{x} = \mathrm{pr}_\beta([\delta]) \in \mathscr{C}\ell_F^+(p^\beta)$, for pr_β as in (5.3).

As above, $\det(\delta_1\delta_2^{-1}) \in \mathbf{A}_F^\times$ is a representative of $\mathbf{x}$. Recall the evaluation map $\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\Omega}$ from §4.2.3, and define a 'Galois evaluation' $\mathrm{Ev}_{\beta,[\delta]}$ as the composition

$$\begin{aligned}\mathrm{Ev}_{\beta,[\delta]} : \mathrm{H}_c^t(S_K, \mathscr{D}_\Omega) \xrightarrow{\mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\Omega}} (\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}} &\xrightarrow{\kappa_\Omega^\beta} \mathcal{D}(\mathscr{U}_\beta, \mathcal{O}_\Omega)\\ &\xrightarrow{\mu \mapsto \delta * \mu} \mathcal{D}(\mathrm{Gal}_p[\mathbf{x}], \mathcal{O}_\Omega).\end{aligned} \tag{6.14}$$

Here the action of δ on μ is by (6.5), the map κ_β^Ω was defined in Definition 6.8, and the target is $\mathcal{D}(\mathrm{Gal}_p[\mathbf{x}], \mathcal{O}_\Omega)$ by Lemma 6.1.

Lemma 6.13. $\mathrm{Ev}_{\beta,[\delta]}$ *is independent of the choice of the representative δ of $[\delta] \in \pi_0(X_\beta)$.*

Proof. Recall $\mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$ is a $H(\mathbf{A})$-module via (6.5), with $H(\mathbf{Q})$ and H_∞° acting trivially. Let

$$\kappa : \mathcal{D}_\Omega \xrightarrow{\kappa_\Omega^\beta} \mathcal{D}(\mathscr{U}_\beta, \mathcal{O}_\Omega) \xrightarrow[\sim]{\iota_*} \mathcal{D}(\mathrm{Gal}[\mathbf{1}_\beta], \mathcal{O}_\Omega) \subset \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$$

denote the composition. From Proposition 4.9, we have a map

$$\mathrm{Ev}_{\beta,[\delta]}^{\mathcal{D}_\Omega,\kappa} : \mathrm{H}_c^t(S_K, \mathcal{D}_\Omega) \to \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega).$$

If $\Phi \in \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)$ then by definition we have

$$\mathrm{Ev}^{\mathcal{D}_\Omega;\kappa}_{\beta,[\delta]}(\Phi) = \delta * [\kappa \circ \mathrm{Ev}^{\mathcal{D}_\Omega}_{\beta,\delta}(\Phi)] = \mathrm{Ev}_{\beta,[\delta]}(\Phi).$$

Then independence of δ follows from Proposition 4.9. □

As in Definition 5.4, let η_0 be any finite order character of $\mathscr{C}\ell^+_F(\mathfrak{m})$. Then define

$$\begin{aligned}\mathrm{Ev}^{\eta_0}_{\beta,\mathbf{x}} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega) &\longrightarrow \mathcal{D}(\mathrm{Gal}_p[\mathbf{x}], \mathcal{O}_\Omega)\\ \Phi &\longmapsto \sum_{[\delta]\in \mathrm{pr}_\beta^{-1}(\mathbf{x})} \eta_0^{-1}\big(\mathrm{pr}_2([\delta])\big)\ \mathrm{Ev}_{\beta,[\delta]}(\Phi).\end{aligned}$$

Using (6.2), we finally obtain an evaluation map

$$\begin{aligned}\mathrm{Ev}^{\eta_0}_\beta := \bigoplus_{\mathbf{x}\in\mathscr{C}\ell^+_F(p^\beta)} \mathrm{Ev}^{\eta_0}_{\beta,\mathbf{x}} : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega) &\longrightarrow \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega) \qquad (6.15)\\ \Phi &\longmapsto \sum_{[\delta]\in\pi_0(X_\beta)} \eta_0^{-1}\big(\mathrm{pr}_2([\delta])\big)\ \times \Big(\delta * \Big[\kappa^\beta_\Omega \circ \mathrm{Ev}^{\mathcal{D}_\Omega}_{\beta,\delta}(\Phi)\Big]\Big).\end{aligned}$$

Remark 6.14. In the notation of Remark 5.5, $\mathrm{Ev}^{\eta_0}_\beta$ is the composition

$$\begin{array}{ccccc} & & \xrightarrow{\ \oplus\mathrm{Ev}_{\beta,[\delta]}\ } & & \\ \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega) & \xrightarrow{\oplus \mathrm{Ev}^{\mathcal{D}_\Omega}_{\beta,\delta}} & \bigoplus_{[\delta]}(\mathcal{D}_\Omega)_{\Gamma_{\beta,\delta}} & \xrightarrow{\delta * \kappa^\beta_\Omega} & \bigoplus_{[\delta]} \mathcal{D}(\mathrm{Gal}_p[\mathrm{pr}_\beta([\delta])], \mathcal{O}_\Omega) \\ & & & & \downarrow{\scriptstyle \Sigma_{\mathbf{x}}\Xi^{\eta_0}_{\mathbf{x}}} \\ & & \xrightarrow{\ \mathrm{Ev}^{\eta_0}_\beta\ } & & \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega), \end{array} \qquad (6.16)$$

where again $\Xi^{\eta_0}_{\mathbf{x}}$ sends a tuple $(m_{[\delta]})_{[\delta]}$ to $\sum_{[\delta]\in\mathrm{pr}_\beta^{-1}(\mathbf{x})} \eta_0^{-1}(\mathrm{pr}_2([\delta])) \times m_{[\delta]}$.

The maps $\mathrm{Ev}^{\eta_0}_\beta$ are functorial in Ω. Let $\lambda \in \Omega$, and let $\mathrm{sp}_\lambda : \mathcal{O}_\Omega \to L$ denote evaluation at λ.

Proposition 6.15. *Let $\beta = (\beta_\mathfrak{p})_{\mathfrak{p}|p}$ with $\beta_\mathfrak{p} > 0$ for each $\mathfrak{p}|p$. We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega) & \xrightarrow{\mathrm{Ev}^{\eta_0}_\beta} & \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega) \\ \downarrow{\scriptstyle \mathrm{sp}_\lambda} & & \downarrow{\scriptstyle \mathrm{sp}_\lambda} \\ \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\lambda) & \xrightarrow{\mathrm{Ev}^{\eta_0}_\beta} & \mathcal{D}(\mathrm{Gal}_p, L). \end{array}$$

Proof. We check that every square in the following diagram is commutative, where the horizontal maps are as in Remark 6.14 (with the middle horizontal maps a composition of two of the maps in that remark) and every vertical map is induced from sp_λ:

$$\begin{array}{ccccccc} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega) & \longrightarrow & \bigoplus_{[\delta]} \mathcal{D}(\mathscr{U}_\beta, \mathcal{O}_\Omega) & \longrightarrow & \bigoplus_{\mathbf{x}} \mathcal{D}(\mathrm{Gal}_p[\mathbf{x}], \mathcal{O}_\Omega) & \longrightarrow & \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\lambda) & \longrightarrow & \bigoplus_{[\delta]} \mathcal{D}(\mathscr{U}_\beta, L) & \longrightarrow & \bigoplus_{\mathbf{x}} \mathcal{D}(\mathrm{Gal}_p[\mathbf{x}], L) & \longrightarrow & \mathcal{D}(\mathrm{Gal}_p, L). \end{array}$$

The first square commutes by Proposition 6.12. The second horizontal arrows are induced by $\delta * -$, and sp_λ is $H(\mathbf{A})$-equivariant, so the second square commutes. The remaining horizontal maps are given by taking linear combinations, which commutes with sp_λ. □

Proposition 6.16. *Let $\beta \in (\mathbf{Z}_{\geqslant 0})_{\mathfrak{p}|p}$ and fix $\mathfrak{p}|p$ in F. Suppose that $\beta_{\mathfrak{q}} > 0$ for each $\mathfrak{q}|p$ and let β' be the tuple defined by $\beta'_{\mathfrak{p}} = \beta_{\mathfrak{p}} + 1$ and $\beta'_{\mathfrak{q}} = \beta_{\mathfrak{q}}$ for each prime $\mathfrak{q}|p$ other than $\mathfrak{p}$. Then*

$$\mathrm{Ev}_{\beta'}^{\eta_0} = \mathrm{Ev}_{\beta}^{\eta_0} \circ U_{\mathfrak{p}}^{\circ} \ : \ \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega}) \longrightarrow \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_{\Omega}).$$

Proof. For each $[\delta] \in \pi_0(X_\beta)$, from Proposition 4.10 we deduce

$$\sum_{[\delta'] \in \mathrm{pr}_{\beta,\mathfrak{p}}^{-1}([\delta])} \mathrm{Ev}_{\beta',[\delta']} = \mathrm{Ev}_{\beta,[\delta]} \circ U_{\mathfrak{p}}^{\circ}.$$

Scaling the left-hand side by $\eta_0^{-1}(\mathrm{pr}_2([\delta]))$ and summing over $[\delta] \in \pi_0(X_\beta)$ gives $\mathrm{Ev}_{\beta'}^{\eta_0}$ (see (6.15)), and doing the same on the right-hand side gives $\mathrm{Ev}_{\beta}^{\eta_0} \circ U_{\mathfrak{p}}^{\circ}$, from which we conclude. □

Definition 6.17. Let $\Phi \in \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega})$, and suppose that for every $\mathfrak{p}|p$, Φ is an eigenclass for $U_{\mathfrak{p}}^{\circ}$ with eigenvalue $\alpha_{\mathfrak{p}}^{\circ} \neq 0$. We define

$$\mu^{\eta_0}(\Phi) := \mathrm{Ev}_{\beta}^{\eta_0}(\Phi)/(\alpha_p^{\circ})^{\beta} \in \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_{\Omega}), \tag{6.17}$$

where β is any tuple such that $\beta_{\mathfrak{p}} > 0$ for each $\mathfrak{p}|p$ and $(\alpha_p^{\circ})^{\beta} := \prod_{\mathfrak{p}|p} (\alpha_{\mathfrak{p}}^{\circ})^{\beta_{\mathfrak{p}}}$. By Proposition 6.16, the distribution $\mu^{\eta_0}(\Phi)$ is independent of the choice of β.

6.4. Interpolation of classical evaluations. Fix $\lambda \in X_0^*(T)$. Via specialisation $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\lambda}) \xrightarrow{r_\lambda} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_{\lambda}^{\vee})$, we now relate $\mathrm{Ev}_{\beta}^{\eta_0}$ from (6.16) to the evaluations $\mathcal{E}_{\chi}^{j,\eta_0}$ of (5.5) as λ varies over Ω, j varies in $\mathrm{Crit}(\lambda)$ and χ varies over finite order characters of conductor p^{β}.

Lemma 6.18. *Let $\Phi \in \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\lambda})$. Let χ be a finite order Hecke character of F of conductor p^{β}, with $\beta_{\mathfrak{p}} > 0$ for all $\mathfrak{p}|p$. For all $j \in \mathrm{Crit}(\lambda)$, we have*

$$\int_{\mathrm{Gal}_p} \chi\ \chi_{\mathrm{cyc}}^j \cdot \mathrm{Ev}_{\beta}^{\eta_0}(\Phi) = \mathcal{E}_{\chi}^{j,\eta_0} \circ r_\lambda(\Phi).$$

Proof. In view of Remarks 6.14 and 5.5, the lemma follows directly from commutativity of the following diagram, since the maps $\mathrm{Ev}_{\beta}^{\eta_0}$ and $\mathcal{E}_{\chi}^{j,\eta_0}$ are respectively the left and right columns:

$$\begin{array}{ccc}
\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\lambda}) & \xrightarrow{\quad r_\lambda \quad} & \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_{\lambda}^{\vee}) \\
\Big\downarrow {\scriptstyle \oplus\left(\kappa_\lambda^{\beta} \circ \mathrm{Ev}_{\beta,\delta}^{\mathcal{D}_\lambda}\right)} & & \Big\downarrow {\scriptstyle \oplus\left(\kappa_{\lambda,j}^{\circ} \circ \mathrm{Ev}_{\beta,\delta}^{V_\lambda^{\vee}}\right)} \\
\bigoplus_{[\delta]} \mathcal{D}(\mathscr{U}_\beta, L) & \xrightarrow{\oplus \int_{\mathscr{U}_\beta} \chi_{\mathrm{cyc}}^j} & \bigoplus_{[\delta]} L \\
\Big\downarrow {\scriptstyle \delta *} & & \Big\downarrow {\scriptstyle \delta *} \\
\bigoplus_{[\delta]} \mathcal{D}(\mathrm{Gal}_p[\mathbf{x}], L) & \xrightarrow{\oplus \int_{\mathrm{Gal}_p[\mathbf{x}]} \chi_{\mathrm{cyc}}^j} & \bigoplus_{[\delta]} L \\
\Big\downarrow {\scriptstyle \oplus \Xi_{\mathbf{x}}^{\eta_0}} & & \Big\downarrow {\scriptstyle \oplus \Xi_{\mathbf{x}}^{\eta_0}} \\
\bigoplus_{\mathbf{x}} \mathcal{D}(\mathrm{Gal}_p[\mathbf{x}], L) & \xrightarrow{\oplus \int_{\mathrm{Gal}_p[\mathbf{x}]} \chi_{\mathrm{cyc}}^j} & \bigoplus_{\mathbf{x}} L \\
\Big\downarrow {\scriptstyle \Sigma} & & \Big\downarrow {\scriptstyle \ell \mapsto \Sigma \chi(\mathbf{x}) \ell_{\mathbf{x}}} \\
\mathcal{D}(\mathrm{Gal}_p, L) & \xrightarrow{\int_{\mathrm{Gal}_p} \chi \chi_{\mathrm{cyc}}^j} & L.
\end{array}$$

Recall $\Xi_{\mathbf{x}}^{\eta_0}$ was defined in (5.5) and all the direct sums are over $[\delta] \in \pi_0(X_\beta)$ or $\mathbf{x} \in \mathscr{C}\!\ell_F^+(p^\beta)$, related by $\mathbf{x} = \mathrm{pr}_\beta(\delta)$. The first square commutes by Proposition 6.12. The second square commutes since for $\mu \in \mathcal{D}(\mathscr{U}_\beta, L)$ we have

$$\int_{\mathrm{Gal}_p[\mathbf{z}]} \chi_{\mathrm{cyc}}^j \cdot \delta * \mu = \chi_{\mathrm{cyc}}\Big(\det(\delta_1)^j \det(\delta_2)^{-\mathsf{w}-j}\Big) \int_{\mathrm{Gal}_p[\mathbf{z}]} \chi_{\mathrm{cyc}}^j \cdot \mu$$
$$= \delta * \int_{\mathrm{Gal}_p[\mathbf{z}]} \chi_{\mathrm{cyc}}^j \cdot \mu,$$

where the action of δ in the left-hand term is (6.4), and in the right-hand term is Definition 5.1. The third square commutes by definition of $\Xi_{\mathbf{x}}^{\eta_0}$, and the fourth square commutes since

$$\int_{\mathrm{Gal}_p} \chi\chi_{\mathrm{cyc}}^j \cdot \mu = \sum_{\mathbf{x}\in \mathscr{C}\ell_F^+(p^\beta)} \chi(\mathbf{x}) \int_{\mathrm{Gal}_p[\mathbf{x}]} \chi_{\mathrm{cyc}}^j \cdot \mu. \qquad \square$$

6.5. Admissibility of $\mu^{\eta_0}(\Phi)$. We now assume $\Omega = \{\lambda\}$ is a single algebraic weight, in which case $\mathcal{O}_\Omega = L$ is a finite extension of $\mathbf{Q}_p$. Let Φ and $\{\alpha_{\mathfrak{p}}^\circ : \mathfrak{p}|p\}$ be as in Definition 6.17, and let

$$\alpha_p^\circ := \prod_{\mathfrak{p}|p} (\alpha_{\mathfrak{p}}^\circ)^{e_{\mathfrak{p}}}, \qquad h_p = v_p(\alpha_p^\circ).$$

We show $\mu^{\eta_0}(\Phi)$ satisfies a growth condition depending on h_p that importantly renders it unique for the very small slope case $h_p < \#\mathrm{Crit}(\lambda)$.

As in [11, §3.4], the space $\mathcal{A}(\mathrm{Gal}_p, L)$ of L-valued locally analytic functions on Gal_p is the direct limit $\varinjlim_m \mathcal{A}_m(\mathrm{Gal}_p, L)$ of the spaces which are analytic on all balls of radius $|p|^{-m}$, and each of these is a Banach L-space with respect to a discretely valued norm $||\cdot||_m$. Dualising, we get a family of norms

$$\begin{aligned} ||\mu||_m &:= \sup_{f\in\mathcal{A}_m(\mathrm{Gal}_p,L)} \tfrac{|\mu(f)|}{||f||_m} \\ &= \sup_{||f||_m\leqslant 1} |\mu(f)| \end{aligned} \tag{6.18}$$

on $\mathcal{D}(\mathrm{Gal}_p, L)$, which thus obtains the structure of a Fréchet module.

Definition 6.19. (See [11, Def. 3.10]). Let $h \in \mathbf{Q}_{\geqslant 0}$. We say $\mu \in \mathcal{D}(\mathrm{Gal}_p, L)$ is *admissible of growth* h if there exists $C \geqslant 0$ such that for each $m \in \mathbf{Z}_{\geqslant 1}$, we have $||\mu||_m \leqslant p^{mh}C$.

Proposition 6.20. *Let Φ be as in Definition 6.17, and $h_p = v_p(\alpha_p^\circ)$. Then $\mu^{\eta_0}(\Phi)$ is admissible of growth h_p.*

Proof. We follow the proof of [11, Prop. 3.11], where this is proved for GL_2. For $m \in \mathbf{Z}_{\geqslant 1}$, put

$$\beta_m = (me_{\mathfrak{p}})_{\mathfrak{p}|p},$$

so that

$$|(\alpha_p^\circ)^{-\beta_m}| = p^{mh_p} \qquad \text{and} \qquad p^{\beta_m}\mathcal{O}_{F,p} = p^m\mathcal{O}_{F,p}.$$

By definition of μ^{η_0}, for $f \in \mathcal{A}_m(\mathrm{Gal}_p, L)$ we have

$$|\mu^{\eta_0}(\Phi)(f)| = p^{mh_p}\Bigg|\sum_{[\delta]\in\pi_0(X_{\beta_m})} \eta_0^{-1}(\mathrm{pr}_2([\delta]))\mathrm{Ev}_{\beta_m,\delta}^{\mathcal{D}_\lambda}(\Phi)\left[v_\lambda^{\beta_m}\left(\delta^{-1} * f\big|_{\mathrm{Gal}_p[\mathrm{pr}_{\beta_m}([\delta])]}\right)\right]\Bigg|. \tag{6.19}$$

By (6.19) it suffices to find C such that for all δ, $m \in \mathbf{Z}_{\geqslant 1}$ and $f \in \mathcal{A}_m(\mathrm{Gal}_p, L)$ with $||f||_m \leqslant 1$, we have

$$\left|\mathrm{Ev}_{\beta_m,\delta}^{\mathcal{D}_\lambda}(\Phi)\left[v_\lambda^{\beta_m}\left(\delta^{-1} * f\big|_{\mathrm{Gal}_p[\mathrm{pr}_{\beta_m}([\delta])]}\right)\right]\right| \leqslant C. \tag{6.20}$$

We also have descriptions $\mathcal{A}_\lambda = \varinjlim_m \mathcal{A}_{\lambda,m}$ and $\mathcal{D}_\lambda = \varprojlim_m \mathcal{D}_{\lambda,m}$ as limits of Banach spaces (see [19, §3.2.2]), and each of the $\mathcal{D}_{\lambda,m}$ are preserved by the action of Δ_p (§3.4 *ibid.*). For every $m \geqslant 1$, restriction from $\mathcal{A}_\lambda$ to $\mathcal{A}_{\lambda,m}$ induces a map $\mathcal{D}_\lambda \to \mathcal{D}_{\lambda,m}$. We let $\mathcal{D}_{\lambda,m}^\circ$ denote the $\mathcal{O}_L$-module of distributions $\mu \in \mathcal{D}_{\lambda,m}$ with $||\mu||_m \leqslant 1$, which is a lattice preserved by the action of Δ_p. Note that rescaling Φ does not affect admissibility (it rescales C); so without loss of generality, we can suppose that the image Φ_1 of Φ in $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\lambda,1})$ is contained in the image of $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}^\circ_{\lambda,1})$, that is, there exists Φ_1° such that we have

$$\begin{array}{ccccc} & & \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}^\circ_{\lambda,1}) & & \Phi_1^\circ \\ & & \downarrow & & \downarrow \\ \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\lambda) & \longrightarrow & \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\lambda,1}) & \Phi \longmapsto & \Phi_1 \end{array}$$

Fix δ and $m \in \mathbf{Z}_{\geqslant 1}$. For ease of notation, let

$$t_m := \xi t_p^{\beta_m}, \qquad \Gamma_m := \Gamma_{\beta_m,\delta}.$$

As in the proof of Lemma 6.2 there exist $\mu \in \mathcal{D}_\lambda$ and $\mu_1 \in \mathcal{D}^\circ_{\lambda,1}$ such that

$$\mathrm{Ev}^{\mathcal{D}_\lambda}_{\beta_m,\delta}(\Phi) = (t_m * \mu)_\delta$$
$$\text{and} \qquad \mathrm{Ev}^{\mathcal{D}_{\lambda,1}}_{\beta_m,\delta}(\Phi_1) = \mathrm{Ev}^{\mathcal{D}^\circ_{\lambda,1}}_{\beta_m,\delta}(\Phi_1^\circ) = (t_m * \mu_1)_\delta,$$

where in the second equation, we have applied Lemma 4.6 with κ the inclusion $\mathcal{D}^\circ_{\lambda,1} \hookrightarrow \mathcal{D}_{\lambda,1}$. By Lemma 4.6 applied again, now with κ the map $\mathcal{D}_\lambda \to \mathcal{D}_{\lambda,1}$, we deduce

$$\mu\Big|_{t_m^{-1} * \mathcal{A}_{\lambda,1}^{\Gamma_m}} = \mu_1\Big|_{t_m^{-1} * \mathcal{A}_{\lambda,1}^{\Gamma_m}}. \tag{6.21}$$

Note if $g \in \mathcal{A}_{\lambda,m}$, then by definition g is analytic on

$$\{\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right) : X \in -I_n + p^m M_n(\mathcal{O}_{F,p})\} \subset N_Q(\mathbf{Z}_p).$$

Since the action of t_m sends $N_Q(\mathbf{Z}_p)$ onto this subset (see (6.9)), we have

$$t_m^{-1} * g \in t_m * \mathcal{A}_{\lambda,m} \subset \mathcal{A}_{\lambda,0} \subset \mathcal{A}_{\lambda,1}$$

(i.e. t_m sends m-analytic functions to analytic functions). As $v_\lambda^{\beta_m}$ preserves m-analyticity, we thus have

$$t_m^{-1} * v_\lambda^{\beta_m}[\mathcal{A}_m(1 + p^m\mathcal{O}_{F,p}, L)] \subset \mathcal{A}_{\lambda,1},$$

and we can evaluate μ_1 on this set. Then:

Claim 6.21. *We have*

$$\mu\Big|_{t_m^{-1} * v_\lambda^{\beta_m}\left[\mathcal{A}_m(1+p^m\mathcal{O}_{F,p},L)\right]} = \mu_1\Big|_{t_m^{-1} * v_\lambda^{\beta_m}\left[\mathcal{A}_m(1+p^m\mathcal{O}_{F,p},L)\right]}. \tag{6.22}$$

We explain how Proposition 6.20 follows from the claim. For f as above, let $f_\delta := \delta^{-1} * f|_{\mathrm{Gal}_p[\mathrm{pr}_{\beta_m}([\delta])]}$. As in Lemma 6.1, we have

$$f_\delta \in \mathcal{A}_m(\mathscr{U}_{\beta_m}, L) \subset \mathcal{A}_m(1 + p^m\mathcal{O}_{F,p}, L).$$

Moreover $||t_m^{-1} * v_\lambda^{\beta_m}(f_\delta)||_1 \leqslant 1$: indeed

– $||f||_m \leqslant 1$ by assumption;

– the action of δ^{-1} preserves integrality (as χ_{cyc} is valued in $\mathbf{Z}_p^\times$);

– $v_\lambda^{\beta_m}$ preserves integrality (as v_λ^H was chosen integral); and

– t_m^{-1} preserves integrality (as it acts only on the argument).

Thus

$$\left|\mathrm{Ev}^{\mathcal{D}_\lambda}_{\beta_m,\delta}(\Phi)\left[v_\lambda^{\beta_m}\left(\delta^{-1} * f\big|_{\mathrm{Gal}_p[\mathrm{pr}_{\beta_m}([\delta])]}\right)\right]\right| = \left|\mu\big(t_m^{-1} * v_\lambda^{\beta_m}(f_\delta)\big)\right|$$
$$= \left|\mu_1\big(t_m^{-1} * v_\lambda^{\beta_m}(f_\delta)\big)\right| \leqslant ||\mu_1||_1 \leqslant 1,$$

where the first equality is by definition, the second is Claim 6.21, the third inequality is by definition of $||\cdot||_1$ on $\mathcal{D}_{\lambda,1}$ (the direct analogue of (6.18)) using $||t_m^{-1} * v_\lambda^{\beta_m}(f_\delta)||_1 \leqslant 1$, and the last inequality follows as $\mu_1 \in \mathcal{D}_{\lambda,1}^\circ$. Since δ, m and f were arbitrary, this shows (6.20) and thus Proposition 6.20.

It remains to prove Claim 6.21. We first motivate the statement, in line with the proof of [11, Prop. 3.11]. We might aim to prove the stronger statement that μ and μ_1 agree on the set $t_m^{-1} * \mathcal{A}_{\lambda,m}^{\Gamma_m}$ (which contains $t_m^{-1} * v_\lambda^{\beta_m}[\mathcal{A}_m(1+p^m\mathcal{O}_{F,p},L)]$); and to do this, it would suffice to show

$$t_m^{-1} * \mathcal{A}_{\lambda,1}^{\Gamma_m} \subset t_m^{-1} * \mathcal{A}_{\lambda,m}^{\Gamma_m}$$

is dense, whence equality would follow from (6.21). However it is not clear how to write down explicit bases of $\mathcal{A}_{\lambda,m}^{\Gamma_m}$. Instead we essentially prove an analogous density for the smaller, but still sufficient, subset in the claim, using explicit bases for $\mathcal{A}_m(1+p^m\mathcal{O}_{F,p},L)$.

We have coordinates $z = (z_\sigma)_{\sigma\in\Sigma}$ on $\mathcal{O}_{F,p}$. Note m-analytic functions on $1+p^m\mathcal{O}_{F,p}$ are analytic, and an orthonormal basis for $\mathcal{A}_m(1+p^m\mathcal{O}_{F,p},L)$ is given by the monomials

$$y_m^i := \left(\frac{z-1}{p^{\beta_m}}\right)^i\Bigg|_{1+p^m\mathcal{O}_{F,p}} = \prod_{\sigma\in\Sigma}\left(\frac{z_\sigma - 1}{\pi_{\mathfrak{p}(\sigma)}^{e_{\mathfrak{p}(\sigma)}m}}\right)^{i_\sigma}\Bigg|_{1+p^m\mathcal{O}_{F,p}}$$

for $i = (i_\sigma) \in \mathbf{N}[\Sigma]$. First we show that for any i, we have

$$\mu\left(t_m^{-1} * v_\lambda^{\beta_m}(y_1^i)\right) = \mu_1\left(t_m^{-1} * v_\lambda^{\beta_m}(y_1^i)\right). \tag{6.23}$$

To see this, note that $v_\lambda^{\beta_1}(y_1^i) \in \mathcal{A}_{\lambda,1}^{\Gamma_1}$ exactly as in Proposition 6.11(iii), and we also have

$$t_m^{-1} * v_\lambda^{\beta_1}(y_1^i) = t_m^{-1} * \left[\left(v_\lambda^{\beta_1}(y_1^i)\right)\Big|_{N_Q^{\beta_m}(\mathbf{Z}_p)}\right]$$
$$= t_m^{-1} * v_\lambda^{\beta_m}(y_1^i),$$

where the first equality follows as the action of t_m sends $N_Q(\mathbf{Z}_p)$ to $N_Q^{\beta_m}(\mathbf{Z}_p)$, and the second from the definition of $v_\lambda^{\beta_m}$. Combining, we have

$$t_m^{-1} * v_\lambda^{\beta_m}(y_1^i) \in t_m^{-1} * \mathcal{A}_{\lambda,1}^{\Gamma_m}$$

and (6.23) follows by (6.21).

Now directly from the definitions we have

$$p^{i(\beta_m-1)}\big[t_m^{-1} * v_\lambda^{\beta_m}(y_m^i)\big] = \big[t_m^{-1} * v_\lambda^{\beta_m}(y_1^i)\big],$$

and combining with (6.23) we deduce

$$\mu\big(t_m^{-1} * v_\lambda^{\beta_m}(y_m^i)\big) = \mu_1\big(t_m^{-1} * v_\lambda^{\beta_m}(y_m^i)\big).$$

Claim 6.21 and Proposition 6.20 follow as the y_m^i are an orthonormal basis of $\mathcal{A}_m(1+p^m\mathcal{O}_{F,p})$. $\square$

6.6. Non-Q-critical p-adic L-functions. We prove Theorem A from the introduction. Let $\tilde{\pi} = (\pi, \{\alpha_{\mathfrak{p}}\}_{\mathfrak{p}|p})$ be a Q-refined RACAR of weight λ satisfying Conditions 2.8′. In particular, it admits an (η, ψ)-Shalika model, with $\eta = \eta_0|\cdot|^{\mathsf{w}}$ and w the purity weight of λ. Suppose that $\tilde{\pi}$ is non-Q-critical (Definition 3.14). Fix $K = K(\tilde{\pi})$ and $\epsilon \in \{\pm 1\}^{\Sigma}$, and let $\phi_{\tilde{\pi}}^{\epsilon} \in \mathrm{H}_{\mathrm{c}}^{t}(S_K, \mathscr{V}_{\lambda}^{\vee})_{\mathfrak{m}_{\tilde{\pi}}}^{\epsilon}$ as in Definition 2.10. By definition of non-Q-criticality, $\phi_{\tilde{\pi}}^{\epsilon}$ lifts uniquely to an eigenclass $\Phi_{\tilde{\pi}}^{\epsilon} \in \mathrm{H}_{\mathrm{c}}^{t}(S_K, \mathscr{D}_{\lambda})_{\mathfrak{m}_{\tilde{\pi}}}^{\epsilon}$ with $U_{\mathfrak{p}}^{\circ}$-eigenvalue $\alpha_{\mathfrak{p}}^{\circ}$, recalling $\alpha_{\mathfrak{p}}^{\circ} = \lambda(t_{\mathfrak{p}})\alpha_{\mathfrak{p}}$. As above, write $\alpha_p^{\circ} = \prod_{\mathfrak{p}|p}(\alpha_{\mathfrak{p}}^{\circ})^{e_{\mathfrak{p}}}$.

Definition 6.22. Let $\mathcal{L}_p^{\epsilon}(\tilde{\pi}) := \mu^{\eta_0}(\Phi_{\tilde{\pi}}^{\epsilon})$ be the distribution on Gal_p attached to $\Phi_{\tilde{\pi}}^{\epsilon}$. Let $\Phi_{\tilde{\pi}} = \sum_{\epsilon \in \{\pm 1\}^{\Sigma}} \Phi_{\tilde{\pi}}^{\epsilon}$, and define the *p-adic L-function* attached to $\tilde{\pi}$ as

$$\begin{aligned}\mathcal{L}_p(\tilde{\pi}) &:= \mu^{\eta_0}(\Phi_{\tilde{\pi}}) \\ &= \sum_{\epsilon \in \{\pm 1\}^{\Sigma}} \mathcal{L}_p^{\epsilon}(\tilde{\pi}) \in \mathcal{D}(\mathrm{Gal}_p, L).\end{aligned}$$

For shorthand, for any $\psi \in \mathcal{A}(\mathrm{Gal}_p, L)$ we write

$$\mathcal{L}_p(\tilde{\pi}, \psi) := \int_{\mathrm{Gal}_p} \psi \cdot \mathcal{L}_p(\tilde{\pi}).$$

Let

$$\mathscr{X}(\mathrm{Gal}_p) := (\mathrm{Spf}\ \mathbf{Z}_p[\![\mathrm{Gal}_p]\!])^{\mathrm{rig}}$$

denote the rigid analytic space of p-adic characters on Gal_p. Via the Amice transform we may view $\mathcal{L}_p(\tilde{\pi}, -) : \mathscr{X}(\mathrm{Gal}_p) \to \overline{\mathbf{Q}}_p$ as an element of $\mathcal{O}(\mathscr{X}(\mathrm{Gal}_p))$.

Theorem 6.23. *The distribution $\mathcal{L}_p(\tilde{\pi})$ is admissible of growth $h_p = v_p(\alpha_p^{\circ})$, and satisfies the following interpolation property: for every finite order Hecke character χ of F of conductor p^{β}, and all $j \in \mathrm{Crit}(\lambda)$, we have*

$$i_p^{-1}(\mathcal{L}_p(\tilde{\pi}, \chi\chi_{\mathrm{cyc}}^j)) = A\tau(\chi_f)^n \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d})^{jn} \prod_{\mathfrak{p}|p} e_{\mathfrak{p}}(\tilde{\pi}, \chi, j) \cdot e_{\infty}(\pi, \chi, j) \cdot \frac{L^{(p)}\big(\pi \otimes \chi, j + \frac{1}{2}\big)}{\Omega_{\pi}^{\epsilon}}, \quad (6.24)$$

where $\epsilon = (\chi\chi_{\mathrm{cyc}}^j\eta)_{\infty}$ and $e_{\infty}(\pi, \chi, j)$ is as in Definition 5.18. At p we have

$$e_{\mathfrak{p}}(\tilde{\pi}, \chi, j) := q_{\mathfrak{p}}^{(\beta_{\mathfrak{p}})\left(nj + \frac{n^2 - n}{2}\right)} \alpha_{\mathfrak{p}}^{-\beta_{\mathfrak{p}}}$$

if $\chi_{\mathfrak{p}}$ is ramified, whilst if $\chi_{\mathfrak{p}}$ is unramified we define

$$e_{\mathfrak{p}}(\tilde{\pi}, \chi, j) := \prod_{i=n+1}^{2n} \frac{1 - \theta_{\mathfrak{p},i}^{-1}\chi_{\mathfrak{p}}^{-1}(\varpi_{\mathfrak{p}})q_{\mathfrak{p}}^{j-1/2}}{1 - \theta_{\mathfrak{p},i}\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})q_{\mathfrak{p}}^{-j-1/2}}.$$

Finally A is the global constant

$$A = \gamma_{p\mathfrak{m}} \cdot \prod_{\mathfrak{p}|p} \frac{q_{\mathfrak{p}}^n}{(q_{\mathfrak{p}} - 1)^n} q_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}\left(\frac{n^2+n}{2}\right)} \in \mathbf{Q}^{\times}. \quad (6.25)$$

Proof. Admissibility is Proposition 6.20. For the interpolation, from Lemma 6.18 we know

$$\int_{\mathrm{Gal}_p} \chi\ \chi_{\mathrm{cyc}}^j \cdot \mu^{\eta_0}(\Phi_{\tilde{\pi}}) = (\alpha_p^{\circ})^{-\beta} \times \mathcal{E}_{\chi,j}^{\eta_0}(\phi_{\tilde{\pi}}),$$

where we must replace $\beta_{\mathfrak{p}}$ with $\max(\beta_{\mathfrak{p}}, 1)$). This is equal to the statement by Theorem 5.22, noting $\lambda(t_p^{\beta})(\alpha_p^{\circ})^{-\beta} = \alpha_p^{-\beta}$ and $\mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d})^{jn} = \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d}^{(p)})^{jn} \prod_{\mathfrak{p}|p} q_{\mathfrak{p}}^{\delta_{\mathfrak{p}} nj}$. Note that $i_p^{-1}(\mathcal{L}_p^{\epsilon}(\tilde{\pi}, \chi\chi_{\mathrm{cyc}}^j)) = 0$ if $\epsilon \neq (\chi\chi_{\mathrm{cyc}}^j\eta)_{\infty}$. ☐

Remarks 6.24. The same theorem holds under the weaker hypothesis of Conditions 2.8′, but with the additional assumption that $\beta_{\mathfrak{p}} \geqslant 1$ for all $\mathfrak{p}|p$, i.e. each $\chi_{\mathfrak{p}}$ is ramified. To include $\beta_{\mathfrak{p}} = 0$ requires a careful analysis of the local zeta integral at $\mathfrak{p}$ for ramified $\pi_{\mathfrak{p}}$ and unramified $\chi_{\mathfrak{p}}$, which was carried out by the second author with Jorza [39].

Finally we consider uniqueness properties of $\mathcal{L}_p(\tilde{\pi})$.

Proposition 6.25. *Suppose Leopoldt's conjecture holds for F at p, and that $h_p < \#\mathrm{Crit}(\lambda)$. Then $\mathcal{L}_p(\tilde{\pi})$ is uniquely determined by its interpolation and admissibility properties.*

Proof. Leopoldt's conjecture implies that Gal_p is 1-dimensional as a p-adic Lie group. Uniqueness is then a result of Vishik [85, Thm. 2.3, Lem. 2.10], shown independently by Amice–Velu [2]. □

When $h_p < \#\mathrm{Crit}(\lambda)$, the restriction of $\mathcal{L}_p(\tilde{\pi})$ to $\mathrm{Gal}_p^{\mathrm{cyc}}$ is unique even without Leopoldt's conjecture. This can be seen by arguments analogous to [11, (78)].

When $h_p \geqslant \#\mathrm{Crit}(\lambda)$, we will prove analogous uniqueness results in §8.5, as an application of our construction of p-adic L-functions in families.

7. Shalika families

For the rest of the paper, we will be concerned with variation in families. In this section, we prove Theorem B of the introduction; namely, we show that: (1) the eigenvariety is étale at a non-Q-critical Q-refined RASCAR $\tilde{\pi}$, and (2) that the unique component through such a $\tilde{\pi}$ is a Shalika family. Since we believe these results to be of independent interest beyond our precise results on p-adic L-functions, we first present them in wide generality here, always working with Hecke operators away from the set S of ramified primes from §2.4. In the process, we develop methods that will be crucially used in the next section, where we make an automorphic hypothesis and add further Hecke operators at each $v \in S$, and refine these results to better suit the study of p-adic L-functions.

Throughout, let $\tilde{\pi}$ be a Q-refined RACAR of weight λ_π satisfying (C1-2) from Conditions 2.8. An undecorated K will always mean an arbitrary subgroup satisfying the conditions of (2.20). We will also consider more specific choices $K(\tilde{\pi}), K_1(\tilde{\pi})$. Unless otherwise specified, we take all coefficients to be in a sufficiently large extension $L/\mathbf{Q}_p$ as in §2.10 and drop it from notation.

7.1. Set-up, statement of Thm. 7.6 and summary of proof.

7.1.1. The eigenvarieties. We introduce local charts around $\tilde{\pi}$ on a parabolic eigenvariety. Fix $h \in \mathbf{Q}_{\geqslant 0}$. Via §3.3, let Ω be an L-affinoid neigbourhood of λ_π in $\mathscr{W}^Q_{\lambda_\pi}$ such that $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)$ admits a slope $\leqslant h$ decomposition with respect to U_p°. Recall $\mathcal{H}$ from §2.9.

Definition 7.1.
- Define $\mathbf{T}_{\Omega,h}(K)$ to be the image of
$$\mathcal{H} \otimes \mathcal{O}_\Omega \longrightarrow \mathrm{End}_{\mathcal{O}_\Omega}\big(\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)^{\leqslant h}\big).$$
- Define
$$\mathscr{E}_{\Omega,h}(K) := \mathrm{Sp}(\mathbf{T}_{\Omega,h}(K)),$$
a rigid analytic space.

Let $w : \mathscr{E}_{\Omega,h}(K) \to \Omega$ be the *weight map* induced by the structure map $\mathcal{O}_\Omega \to \mathbf{T}_{\Omega,h}(K)$. For any $\epsilon \in \{\pm 1\}^\Sigma$, write $\mathbf{T}^\epsilon_{\Omega,h}(K)$ and $\mathscr{E}^\epsilon_{\Omega,h}(K)$ for the analogues using ϵ-parts of the cohomology. As $\mathbf{T}^\epsilon_{\Omega,h}$ is a quotient of $\mathbf{T}_{\Omega,h}$, each $\mathscr{E}^\epsilon_{\Omega,h}(K)$ embeds as a closed subvariety of $\mathscr{E}_{\Omega,h}(K)$. Moreover
$$\mathscr{E}_{\Omega,h}(K) = \textstyle\bigcup_\epsilon \mathscr{E}^\epsilon_{\Omega,h}(K).$$

The local piece $\mathscr{E}_{\Omega,h}(K)$ is the space denoted $\mathcal{E}_{\Omega,h}^{Q,t}$ in [19, §5]. By definition, $\mathscr{E}_{\Omega,h}(K)$ is a rigid space whose L-points y are in bijection with non-trivial algebra homomorphisms $\mathbf{T}_{\Omega,h}(K) \to L$, or equivalently, with systems of eigenvalues $\psi_y : \mathcal{H} \to L$ appearing in $\mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega)^{\leqslant h}$.

We use the convention that $\mathscr{C}$ (resp. $\mathscr{I}$) denotes a connected (resp. irreducible) component of $\mathscr{E}$ (with appropriate decorations).

Definition 7.2. (i) We say a point $y \in \mathscr{E}_{\Omega,h}(K)$ is *classical* if there exists a cohomological automorphic representation π_y of $G(\mathbf{A})$ having weight $\lambda_y := w(y)$ such that ψ_y appears in π_y^K, whence $\tilde{\pi}_y = (\pi_y, \{\psi_y(U_{\mathfrak{p}}^\circ)\}_{\mathfrak{p}|p})$ is a Q-refined automorphic representation. Throughout we use the notation $\mathfrak{m}_y = \mathfrak{m}_{\tilde{\pi}_y}$ for the associated maximal ideal of $\mathbf{T}_{\Omega,h}(K)$.

(ii) A classical point y is *cuspidal* (resp. *essentially self-dual*) if π_y is.

(iii) For a finite order Hecke character η_0, an (η_0, ψ)*-Shalika point* is a classical cuspidal point y such that π_y admits an $(\eta_0|\cdot|^{\mathsf{w}_y}, \psi)$-Shalika model, where w_y is the purity weight of λ_y.

(iv) A *classical (resp. (η_0, ψ)-Shalika) family* in $\mathscr{E}_{\Omega,h}(K)$ is an irreducible component $\mathscr{I}$ in $\mathscr{E}_{\Omega,h}(K)$ containing a Zariski-dense set of classical (resp. (η_0, ψ)-Shalika) points.

To describe the geometry of $\mathscr{E}_{\Omega,h}(K)$, we must be precise about the level K. In Theorem 7.6, there will be two particularly important level groups: the group $K(\tilde{\pi})$ from (2.23), at which Friedberg–Jacquet test vectors exist; and a more explicit group $K_1(\tilde{\pi})$, which we now define. For any place v and $m \in \mathbf{Z}_{\geqslant 0}$ let

$$K_{1,v}(m) \subset \mathrm{GL}_{2n}(\mathcal{O}_v)$$

be the open compact subgroup of matrices whose bottom row is congruent to $(0, ..., 0, 1) \bmod \varpi_v^m$. The *Whittaker conductor* $m(\pi_v)$ of π_v is the minimal integer m such that $\pi_v^{K_{1,v}(m)} \neq 0$, and by [51, §5]

$$\dim_{\mathbf{C}} \ \pi_v^{K_{1,v}(m(\pi_v))} = 1. \tag{7.1}$$

Note $K_{1,v}(0) = \mathrm{GL}_{2n}(\mathcal{O}_v)$, so π_v is spherical if and only if $m(\pi_v) = 0$. We define

$$K_1(\tilde{\pi}) := \textstyle\prod_{\mathfrak{p}|p} J_{\mathfrak{p}} \prod_{v \nmid p} K_{1,v}(m(\pi_v)) \subset G(\mathbf{A}_f). \tag{7.2}$$

7.1.2. Hypotheses on π. Our results require hypotheses on π that we now make precise.

Definition 7.3. We say π *admits a non-zero Deligne-critical L-value at p* if there exists a pair (χ, j) such that

$$L(\pi \otimes \chi, j + \tfrac{1}{2}) \neq 0,$$

where $j \in \mathrm{Crit}(\lambda_\pi)$ and χ is a finite order Hecke character of conductor p^β with $\beta_{\mathfrak{p}} \geqslant 1$ for all $\mathfrak{p}$. This L-value has *sign* ϵ if $\epsilon = (\chi\chi_{\mathrm{cyc}}^j\eta)_\infty \in \{\pm 1\}^\Sigma$.

Note that $L(\pi \otimes \chi, s) \neq 0 \iff L^{(p)}(\pi \otimes \chi, s) \neq 0$ (as the local factors at p are non-vanishing).

Conjecturally, this non-vanishing is true for all but finitely many such pairs (χ, j), so every π should satisfy this hypothesis. In practice, this is guaranteed by the following simple criterion.

Lemma 7.4. *π has a non-zero Deligne-critical L-value at p if $(\lambda_\pi)_{\sigma,n} > (\lambda_\pi)_{\sigma,n+1}$ $\forall \sigma \in \Sigma$.*

Proof. Let j be the largest integer in $\mathrm{Crit}(\lambda_\pi)$, and χ any Hecke character satisfying the conditions of Definition 7.3. The hypothesis ensures that $\#\mathrm{Crit}(\lambda_\pi) > 1$, so that $j + \frac{1}{2} \geqslant \frac{\mathsf{w}}{2} + 1$ (recalling w is the purity weight), and hence $L(\pi \otimes \chi, j + \frac{1}{2}) \neq 0$ by the main result of [52]. ☐

Definition 7.5. Recall λ is regular if $\lambda_{\sigma,i} > \lambda_{\sigma,i+1}$ for all σ and i. Say it is *H-regular* if

$$\lambda_{\sigma,1} > \cdots > \lambda_{\sigma,n} \quad \text{and} \quad \lambda_{\sigma,n+1} > \cdots > \lambda_{\sigma,2n} \tag{7.3}$$

for all σ (allowing $\lambda_{\sigma,n} = \lambda_{\sigma,n+1}$). Such weights are regular as weights for H.

For a field E, let $\mathrm{G}_E := \mathrm{Gal}(\overline{E}/E)$. Attached to any RACAR π' of $G(\mathbf{A})$ we have a Galois representation $\rho_{\pi'} : \mathrm{G}_F \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p)$, depending on our fixed isomorphism $\iota_p : \mathbf{C} \cong \overline{\mathbf{Q}}_p$ (see [47]). For a finite prime v of F, we say *Local-Global Compatibility holds for π' at v* if

$$\mathrm{WD}(\rho_{\pi'}|_{\mathrm{G}_{F_v}})^{\mathrm{F-ss}} = \iota_p \mathrm{rec}_{F_v}(\pi'_v \otimes |\cdot|^{(1-n)/2}),$$

where rec_{F_v} denotes the local Langlands correspondence for GL_{2n}/F_v. This is conjecturally always true; it is known in general up to semi-simplification [82], and is known when π' is essentially self-dual (for self-dual RACARs this is shown in [76, 29]; it is explained in [34, §4.3] why the essentially self-dual case follows). Hence it is known if π' is a RASCAR.

7.1.3. Statement. Let $\tilde{\pi}$ be as in Conditions 2.8, of weight λ_π, and let $\alpha_p^\circ = \prod_{\mathfrak{p}|p}(\alpha_{\mathfrak{p}}^\circ)^{e_{\mathfrak{p}}}$. Recall $K(\tilde{\pi})$ from (2.23) and $\eta = \eta_0|\cdot|^{\mathsf{w}}$ from §2.6. Fix $h \geqslant v_p(\alpha_p^\circ)$. In the rest of §7, we will prove:

Theorem 7.6. *(a) If $\tilde{\pi}$ is strongly non-Q-critical at p (see Definition 3.14), then for any K as in* (2.20) *there is a point $x_{\tilde{\pi}}(K) \in \mathscr{E}_{\Omega,h}(K)$ attached to $\tilde{\pi}$.*

(b) Suppose further that π admits a non-zero Deligne-critical L-value at p. At level $K(\tilde{\pi})$, there exists an irreducible component in $\mathscr{E}_{\Omega,h}(K(\tilde{\pi}))$ through $x_{\tilde{\pi}}(K(\tilde{\pi}))$ of dimension $\dim(\Omega)$.

(c) Suppose further that λ_π is H-regular. There exists an (η_0, ψ)-Shalika family $\mathscr{I}(K(\tilde{\pi}))$ in $\mathscr{E}_{\Omega,h}(K(\tilde{\pi}))$ of dimension $\dim(\Omega)$.

(d) Suppose further that $\rho_\pi : \mathrm{G}_F \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p)$ is irreducible. Then:

(d1) at level $K_1(\tilde{\pi})$, $\mathscr{E}_{\Omega,h}(K_1(\tilde{\pi}))$ is étale over Ω at $x_{\tilde{\pi}}(K_1(\tilde{\pi}))$, and the (irreducible) connected component $\mathscr{C}(K_1(\tilde{\pi}))$ through $x_{\tilde{\pi}}(K_1(\tilde{\pi}))$ is an (η_0, ψ)-Shalika family;

(d2) at level $K(\tilde{\pi})$, $\mathscr{I}(K(\tilde{\pi}))$ is the unique Shalika family of $\mathscr{E}_{\Omega,h}(K(\tilde{\pi}))$ through $x_{\tilde{\pi}}(K(\tilde{\pi}))$. Moreover the nilreduction of $\mathscr{I}(K(\tilde{\pi}))$ is étale over Ω at $x_{\tilde{\pi}}(K(\tilde{\pi}))$.

(e) Suppose further that Local-Global Compatibility holds at all $v \nmid p$ for all RACARs of G. Then in (d2), $\mathscr{I}(K(\tilde{\pi}))$ is also the unique classical family of $\mathscr{E}_{\Omega,h}(K(\tilde{\pi}))$ through $x_{\tilde{\pi}}(K(\tilde{\pi}))$.

It is important to be precise about the level K, so we take a maximal (if unwieldy) approach to notation. If K is completely unambiguous we will drop it from notation.

Remark 7.7. Theorem B of the introduction is a special case of (d1). Indeed, non-Q-critical slope implies strongly non-Q-critical (Theorem 3.16), and if λ_π is regular then it is H-regular (by definition) and π admits a non-zero Deligne-critical L-value at p (Lemma 7.4), hence $\mathcal{L}_p(\tilde{\pi}) \neq 0$. Conjecturally, if π is cuspidal then ρ_π is always irreducible.

For the convenience of the reader we summarise the key steps of the proof.

- We first show that specialisation $\mathrm{sp}_{\lambda_\pi} : \mathcal{D}_\Omega \to \mathcal{D}_{\lambda_\pi}$ induces an isomorphism on the $\tilde{\pi}$-part of degree t cohomology (Proposition 7.8). The existence of $x_{\tilde{\pi}}(K)$ follows immediately. Here we crucially use that t is the *top* degree of cohomology in which $\tilde{\pi}$ appears.

- From §6, for each β we have a $\mathcal{O}_\Omega$-module map $\mathrm{Ev}_\beta^{\eta_0} : \mathrm{H}^t_c(S_K, \mathscr{D}_\Omega) \to \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$, and the target is torsion-free. When $K = K(\tilde{\pi})$, we show that this map is non-zero for some β using Proposition 7.8, the non-vanishing L-value and Theorem 5.22. More generally, the map is non-zero if $\mathcal{L}_p(\tilde{\pi}) \neq 0$. Given non-vanishing, we deduce in Corollary 7.12 that $\mathbf{T}_{\Omega,h}(K(\tilde{\pi}))$ is faithful over $\mathcal{O}_\Omega$ locally at $x_{\tilde{\pi}}(K(\tilde{\pi}))$, and deduce (b) from this.

- Using evaluation maps and (again) the non-vanishing L-value, we construct an everywhere non-vanishing rigid function on Ω, whose value at each classical λ is a sum of Friedberg–Jacquet integrals over the finite set $y \in w^{-1}(\lambda) \subset \mathscr{E}_{\Omega,h}(K(\tilde{\pi}))$. In Proposition 7.16, we use this and Proposition 5.15 to deduce (c).

- We can produce dimension in $\mathscr{E}_{\Omega,h}$ at level $K(\tilde\pi)$, but cannot control the size of the classical cohomology at this level. However, at level $K_1(\tilde\pi)$, by (7.1) $\mathrm{H}^t_{\mathrm{c}}(S_{K_1(\tilde\pi)}, \mathscr{V}^\vee_{\lambda_\pi})^\epsilon_{\mathfrak{m}_{\tilde\pi}}$ is a line. Using commutative algebra we deduce $\mathbf{T}^\epsilon_{\Omega,h}(K_1(\tilde\pi))$ is cyclic locally at $x_{\tilde\pi}(K_1(\tilde\pi))$. Via Local-Global Compatibility for RASCARs and p-adic Langlands functoriality, we prove a 'level-shifting' result between levels $K_1(\tilde\pi)$ and $K(\tilde\pi)$ in families, giving a precise compatibility between $\mathscr{E}_{\Omega,h}(K(\tilde\pi))$ and $\mathscr{E}^\epsilon_{\Omega,h}(K_1(\tilde\pi))$ at $\tilde\pi$. Combining cyclicity at level $K_1(\tilde\pi)$ with faithfulness at level $K(\tilde\pi)$, we deduce (d). Part (e) follows similarly.

7.2. Proof of Thm. 7.6(a): Existence of $x_{\tilde\pi}(K)$. Recall t, from (2.6), is the top degree of classical cohomology to which π contributes. If $\tilde\pi$ is non-Q-critical, then by Theorem 3.16

$$\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\lambda_\pi})_{\mathfrak{m}_{\tilde\pi}} \cong \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{V}_{\lambda_\pi})_{\mathfrak{m}_{\tilde\pi}},$$

which does not vanish as $\pi_f^K \neq 0$. Thus $\tilde\pi$ contributes to $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\lambda_\pi})$. The character $\psi_{\tilde\pi}$: $\mathcal{H}\otimes E \to E$ from Definition 2.9 induces a character $\mathcal{H}\otimes\mathcal{O}_\Omega \to \mathcal{O}_\Omega$, and thus a map

$$\mathcal{H}\otimes\mathcal{O}_\Omega \longrightarrow \mathcal{O}_\Omega \longrightarrow \mathcal{O}_\Omega/\mathfrak{m}_{\lambda_\pi} = L, \tag{7.4}$$

where $\mathfrak{m}_{\lambda_\pi}$ is the maximal ideal corresponding to λ_π. We also write $\mathfrak{m}_{\tilde\pi}$ for the kernel of this composition. This is a maximal ideal of $\mathcal{H}\otimes\mathcal{O}_\Omega$, whose contraction to $\mathcal{O}_\Omega$ is $\mathfrak{m}_{\lambda_\pi}$.

For any sufficiently large h, the localisation $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)^{\leqslant h}_{\mathfrak{m}_{\tilde\pi}}$ is independent of h, and in a slight abuse of notation, we denote this $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)_{\mathfrak{m}_{\tilde\pi}}$. As $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)^{\leqslant h}$ is finitely generated, we may freely use Nakayama's lemma.

Let

$$\mathbf{T}_{\Omega,\tilde\pi}(K) = [\mathbf{T}_{\Omega,h}(K)]_{\mathfrak{m}_{\tilde\pi}}$$

be the localisation of $\mathbf{T}_{\Omega,h}(K)$ at $\mathfrak{m}_{\tilde\pi}$, which acts on $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)_{\mathfrak{m}_{\tilde\pi}}$. Let Λ denote the localisation of $\mathcal{O}_\Omega$ at $\mathfrak{m}_{\lambda_\pi}$. Theorem 7.6(a) follows from:

Proposition 7.8. *The map $\mathrm{sp}_{\lambda_\pi} : \mathcal{D}_\Omega \to \mathcal{D}_{\lambda_\pi}$ induces an isomorphism*

$$\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)_{\mathfrak{m}_{\tilde\pi}} \otimes_\Lambda \Lambda/\mathfrak{m}_{\lambda_\pi} \xrightarrow{\sim} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\lambda_\pi})_{\mathfrak{m}_{\tilde\pi}}. \tag{7.5}$$

In particular, $\mathfrak{m}_{\tilde\pi}$ is a maximal ideal of $\mathbf{T}_{\Omega,h}(K)$ and hence there exists a point $x_{\tilde\pi} \in \mathscr{E}_{\Omega,h}(K)$.

Proof. As $\Omega \subset \mathscr{W}^Q_{\lambda_\pi}$ is smooth, we may choose a regular sequence of generators $T_1,\dots,T_m$ of $\mathfrak{m}_{\lambda_\pi}$. For $j = 1,\dots,m$ let

$$\Omega_j \coloneqq \mathrm{Sp}(\mathcal{O}_\Omega/(T_1,...,T_j));$$

then

$$\Omega = \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_m = \{\lambda_\pi\}$$

is a strictly descending sequence of closed affinoid subsets of Ω containing λ_π. Let Λ_j be the localisation of $\mathcal{O}_{\Omega_j}$ at $\mathfrak{m}_{\lambda_\pi}$, noting that $\Lambda_0 = \Lambda$ and $\Lambda_m = L$. We first prove a vanishing result.

Lemma. *For any $i \geqslant t+1$, and any $j = 0,\dots,m$, we have*

$$\mathrm{H}^i_{\mathrm{c}}(S, \mathscr{D}_{\Omega_j})_{\mathfrak{m}_{\tilde\pi}} = 0.$$

Proof of lemma. We proceed by descending induction on j. The case $j = m$ follows from non-Q-criticality; indeed if $\mathrm{H}^i_{\mathrm{c}}(S, \mathscr{D}_{\lambda_\pi})_{\mathfrak{m}_{\tilde\pi}} \neq 0$, then $\tilde\pi$ appears in classical cohomology in degree i from Definition 3.14. But $i > t$ is greater than the top such degree (see (2.6)), which is a contradiction.

Now suppose the lemma holds for $j+1$. As

$$\mathcal{D}_{\Omega_j}/T_{j+1}\mathcal{D}_{\Omega_j} = \mathcal{D}_{\Omega_j} \otimes_{\mathcal{O}_{\Omega_j}} \mathcal{O}_{\Omega_{j+1}} \cong \mathcal{D}_{\Omega_{j+1}},$$

where the last isomorphism is Remark 3.12, we have a short exact sequence

$$0 \to \mathcal{D}_{\Omega_j} \xrightarrow{\times T_{j+1}} \mathcal{D}_{\Omega_j} \longrightarrow \mathcal{D}_{\Omega_{j+1}} \to 0, \tag{7.6}$$

yielding a long exact sequence of cohomology. We pass to small slope subspaces and then localise at $\mathfrak{m}_{\tilde{\pi}}$; since these are exact functors, truncating at degree i we get an injection

$$0 \to \mathrm{H}^i_{\mathrm{c}}(S, \mathscr{D}_{\Omega_j})_{\mathfrak{m}_{\tilde{\pi}}} \otimes_{\Lambda_j} \Lambda_j/(T_{j+1}) \longrightarrow \mathrm{H}^i_{\mathrm{c}}(S, \mathscr{D}_{\Omega_{j+1}})_{\mathfrak{m}_{\tilde{\pi}}}.$$

By the induction step, we deduce that $\mathrm{H}^i_{\mathrm{c}}(S, \mathscr{D}_{\Omega_j})_{\mathfrak{m}_{\tilde{\pi}}} \otimes_{\Lambda_j} \Lambda_j/(T_{j+1}) = 0$. From Nakayama's lemma, the result follows for j, completing the induction. □

We return to the proof of Proposition 7.8. Let $j \in \{0, \ldots, m-1\}$. Again truncating and localising the long exact sequence of cohomology attached to (7.6), we obtain an exact sequence

$$0 \to \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega_j})_{\mathfrak{m}_{\tilde{\pi}}} \otimes_{\Lambda_j} \Lambda_j/(T_{j+1}) \longrightarrow \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega_{j+1}})_{\mathfrak{m}_{\tilde{\pi}}} \longrightarrow \mathrm{H}^{t+1}_{\mathrm{c}}(S, \mathscr{D}_{\Omega_j})_{\mathfrak{m}_{\tilde{\pi}}}.$$

But the last term is zero by the lemma, showing there is an isomorphism

$$\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega_j})_{\mathfrak{m}_{\tilde{\pi}}} \otimes_{\Lambda_j} \Lambda_j/(T_{j+1}) \cong \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega_{j+1}})_{\mathfrak{m}_{\tilde{\pi}}}.$$

The isomorphism (7.5) follows from descending induction on j.

For the last claim, we have $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega})_{\mathfrak{m}_{\tilde{\pi}}} \neq 0$ by combining non-Q-criticality and (7.5). This is equivalent to $\mathfrak{m}_{\tilde{\pi}}$ appearing in $\mathbf{T}_{\Omega,h}(K)$, from which we deduce the existence of the point $x_{\tilde{\pi}}(K)$.

□

We would also like an analogue of [11, Lem. 2.9(ii)], to show that $\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega})_{\mathfrak{m}_{\tilde{\pi}}}$ is $\mathcal{O}_{\Omega}$-torsion-free. However the proof of that result does *not* work here, as the cohomology is not concentrated in one degree. A key novelty of this paper is the use of evaluation maps to overcome this.

Remark 7.9. All we used to prove Proposition 7.8 was non-Q-criticality and cohomological vanishing above the top degree t. Thus for *any* non-Q-critical Hecke eigensystem attached to a Q-refined RACAR π' of $G(\mathbf{A})$ of weight λ, we see that sp_λ induces an isomorphism

$$\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega})_{\mathfrak{m}_{\tilde{\pi}'}} \otimes_{\mathcal{O}_{\Omega,\lambda}} \mathcal{O}_{\Omega,\lambda}/\mathfrak{m}_\lambda \xrightarrow{\sim} \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\lambda})_{\mathfrak{m}_{\tilde{\pi}'}}.$$

7.3. Proof of Thm. 7.6(b): Components of maximal dimension.

Let

$$\mathscr{C}(K) = \mathrm{Sp}(\mathbf{T}_{\Omega,\mathscr{C}}(K)) \subset \mathscr{E}_{\Omega,h}(K)$$

be the connected component containing $x_{\tilde{\pi}}(K)$. There exists an idempotent e such that $\mathbf{T}_{\Omega,\mathscr{C}}(K) = e\mathbf{T}_{\Omega,h}(K)$ is a direct summand, with $\mathscr{C} = \mathrm{Sp}(\mathbf{T}_{\Omega,\mathscr{C}}(K))$; then

$$\begin{aligned}\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega})^{\leqslant h} \otimes_{\mathbf{T}_{\Omega,h}(K)} \mathbf{T}_{\Omega,\mathscr{C}}(K) &= e\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega})^{\leqslant h} \\ &\subset \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_{\Omega})^{\leqslant h}.\end{aligned} \tag{7.7}$$

Now fix $K = K(\tilde{\pi})$ from (2.23) and for convenience let

$$\mathscr{C} = \mathscr{C}(K(\tilde{\pi})), \qquad \mathbf{T}_{\Omega,\mathscr{C}} = \mathbf{T}_{\Omega,\mathscr{C}}(K(\tilde{\pi})) \qquad \text{and} \qquad x_{\tilde{\pi}} = x_{\tilde{\pi}}(K(\tilde{\pi})).$$

Let $\Phi_{\tilde{\pi}} \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_{\lambda_\pi})_{\mathfrak{m}_{\tilde{\pi}}}$ be the class from Definition 6.22. By Proposition 7.8, we can lift this to a class $\Phi'_{\mathscr{C}} \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_{\Omega})^{\leqslant h}_{\mathfrak{m}_{\tilde{\pi}}}$ under the natural surjection $\mathrm{sp}_{\lambda_\pi}$. Possibly shrinking Ω

and $\mathscr{C}$, we may avoid denominators in $\mathbf{T}_{\Omega,\mathscr{C}}$, and assume $\Phi'_{\mathscr{C}} \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h}$; then applying the idempotent e attached $\mathscr{C}$, we define

$$\Phi_{\mathscr{C}} := e\Phi'_{\mathscr{C}} \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h} \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega,\mathscr{C}}. \tag{7.8}$$

As shrinking Ω and applying e doesn't change local behaviour at $\tilde{\pi}$, we still have $\mathrm{sp}_{\lambda_\pi}(\Phi_{\mathscr{C}}) = \Phi_{\tilde{\pi}}$.

The following is the key step in all our constructions; we are very grateful to Eric Urban, who suggested the elegant proof we present here. Recall $\mathrm{Ev}_\beta^{\eta_0}$ from (6.15).

Proposition 7.10. *Suppose there exists β such that $\mathrm{Ev}_\beta^{\eta_0}(\Phi_{\tilde{\pi}}) \neq 0$. Then $\mathrm{Ann}_{\mathcal{O}_\Omega}(\Phi_{\mathscr{C}}) = 0$, and in particular,*

$$\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h} \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega,\mathscr{C}} \text{ is a faithful } \mathcal{O}_\Omega\text{-module.}$$

Proof. By restricting the evaluation map of (6.15) to the summand (7.7), we get a map

$$\mathrm{Ev}_\beta^{\eta_0} : \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h} \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega,\mathscr{C}} \to \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega) \tag{7.9}$$

of $\mathcal{O}_\Omega$-modules. From Proposition 6.15, we have

$$\mathrm{sp}_{\lambda_\pi}(\mathrm{Ev}_\beta^{\eta_0}(\Phi_{\mathscr{C}})) = \mathrm{Ev}_\beta^{\eta_0}(\mathrm{sp}_{\lambda_\pi}(\Phi_{\mathscr{C}})) = \mathrm{Ev}_\beta^{\eta_0}(\Phi_{\tilde{\pi}}).$$

The right-hand side is non-zero by hypothesis, so we deduce that $\mathrm{Ev}_\beta^{\eta_0}(\Phi_{\mathscr{C}}) \neq 0$.

Now let $u \in \mathcal{O}_\Omega$ such that $u\Phi_{\mathscr{C}} = 0$. Since $\mathrm{Ev}_\beta^{\eta_0}$ is an $\mathcal{O}_\Omega$-module map, we see

$$\begin{aligned} 0 &= \mathrm{Ev}_\beta^{\eta_0}(u\Phi_{\mathscr{C}}) \\ &= u\mathrm{Ev}_\beta^{\eta_0}(\Phi_{\mathscr{C}}) \in \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega). \end{aligned}$$

As $\mathrm{Ev}_\beta^{\eta_0}(\Phi_{\mathscr{C}}) \neq 0$ and $\mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$ is $\mathcal{O}_\Omega$-torsion-free, we see $u = 0$, from which we conclude. ☐

Corollary 7.11. *Suppose there exists β such that $\mathrm{Ev}_\beta^{\eta_0}(\Phi_{\tilde{\pi}}) \neq 0$. Then, at level $K(\tilde{\pi})$,*

(i) the $\mathcal{O}_\Omega$-algebra $\mathbf{T}_{\Omega,\mathscr{C}}$ is faithful as an $\mathcal{O}_\Omega$-module, and

(ii) there exists an irreducible component $\mathscr{I} \subset \mathscr{C}$ in $\mathscr{E}_{\Omega,h}$ through $x_{\tilde{\pi}}$ with $\dim(\mathscr{I}) = \dim(\Omega)$.

Proof. The $\mathcal{O}_\Omega$-action on $\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h}$ is faithful (by Proposition 7.10) and factors through the action of $\mathcal{H} \otimes \mathcal{O}_\Omega$, hence (by definition) the action of $\mathbf{T}_{\Omega,\mathscr{C}}$. Part (i) follows.

For (ii), as $\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h}$ and $\mathbf{T}_{\Omega,h}$ are finite $\mathcal{O}_\Omega$-modules, we deduce there are finitely many irreducible components of $\mathcal{E}_{\Omega,h}$, each of dimension at most $\dim(\Omega)$. Suppose every component $\mathscr{I}$ through $x_{\tilde{\pi}}$ has dimension $\dim(\mathscr{I}) < \dim(\Omega)$. Then $\mathrm{Supp}_{\mathcal{O}_\Omega}(\mathbf{T}_{\Omega,\mathscr{C}})$ (by definition, the image of $\mathscr{C}$ in Ω under the weight map) is a closed subspace of Ω of dimension strictly less than $\dim(\Omega)$. In particular it is a *proper* closed subspace. But by [46, Prop. 4.4.2], since $\mathbf{T}_{\Omega,\mathscr{C}}$ is a faithful $\mathcal{O}_\Omega$-module, we have $\mathrm{Supp}_{\mathcal{O}_\Omega}(\mathbf{T}_{\Omega,\mathscr{C}}) = \Omega$, so we conclude by contradiction. ☐

Corollary 7.12. *Suppose π admits a non-zero Deligne-critical L-value at p. Then (at level $K(\tilde{\pi})$) there exists an irreducible component $\mathscr{I}$ in $\mathscr{E}_{\Omega,h}$ through $x_{\tilde{\pi}}$ such that $\dim(\mathscr{I}) = \dim(\Omega)$.*

Proof. By hypothesis (Definition 7.3) there exists β with $\beta_{\mathfrak{p}} \geqslant 1$ for all $\mathfrak{p}|p$, a character χ of conductor p^β, and $j \in \mathrm{Crit}(\lambda_\pi)$ such that $L(\pi \times \chi, j + \frac{1}{2}) \neq 0$. By Theorem 6.23, for an explicit (non-zero) constant $(*)$ we have

$$\begin{aligned} (\alpha_p^\circ)^{-\beta} \int_{\mathrm{Gal}_p} \chi\, \chi_{\mathrm{cyc}}^j \cdot \mathrm{Ev}_\beta^{\eta_0}(\Phi_{\tilde{\pi}}) &=: \mathcal{L}_p(\tilde{\pi}, \chi\chi_{\mathrm{cyc}}^j) \\ &= (*)L^{(p)}(\pi \otimes \chi, j + \tfrac{1}{2}) \neq 0. \end{aligned}$$

Thus $\mathrm{Ev}_\beta^{\eta_0}(\Phi_{\tilde{\pi}}) \neq 0$. We conclude by Corollary 7.11(ii). ☐

7.4. Proof of Thm. 7.6(c): (very) Zariski-density of Shalika points. We still take $K = K(\tilde{\pi})$.

Lemma 7.13. *If λ_π is H-regular in the sense of (7.3), then any neighbourhood Ω of λ_π in $\mathscr{W}^Q_{\lambda_\pi}$ contains a very Zariski-dense set of regular algebraic dominant weights.*

Proof. Write $\lambda_\pi = (\lambda'_\pi, \lambda''_\pi)$ as a weight for H. If a weight $\lambda \in \mathscr{W}^Q_{\lambda_\pi}$ is of the form

$$\lambda = (\lambda'_\pi + (a, ...a), \lambda''_\pi + (b, ..., b)),$$

where $a, b \in \mathbf{Z}^\Sigma$ are weights for $\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}(\mathrm{GL}_1)$ with $a_\sigma \geqslant b_\sigma$ for all $\sigma \in \Sigma$, then λ is algebraic dominant and H-regular. The set of such λ is very Zariski-dense in $\mathscr{W}^Q_{\lambda_\pi}$.

Moreover, such a weight is regular if $a_\sigma > b_\sigma$ for all σ, as then $\lambda_{\pi,n,\sigma} + a_\sigma > \lambda_{\pi,n+1,\sigma} + b_\sigma$. We conclude since $a_\sigma = b_\sigma$ is a closed condition. □

Recall we fixed $h \in \mathbf{Q}_{\geqslant 0}$. Now let Ω_{ncs} be the subset of weights $\lambda \in \Omega$ such that

(i) λ is algebraic, dominant and regular, and

(ii) $e_{\mathfrak{p}(\sigma)}h < 1 + \lambda_{\sigma,n} - \lambda_{\sigma,n+1}$ for all $\sigma \in \Sigma$ (in particular, h is a non-Q-critical slope for λ).

Since failure of (ii) is a closed condition, Ω_{ncs} is very Zariski-dense in Ω by Lemma 7.13.

Proposition 7.14. *Suppose $\tilde{\pi}$ is strongly non-Q-critical and λ_π is H-regular. Let $\mathscr{I} = \mathrm{Sp}(\mathbf{T}_{\Omega,\mathscr{I}})$ be any irreducible component in $\mathscr{E}_{\Omega,h}(K(\tilde{\pi}))$ such that*

- *$\mathcal{I}$ contains $x_{\tilde{\pi}}(K(\tilde{\pi}))$, and*
- *$\dim(\mathscr{I}) = \dim(\Omega)$.*

Then the classical cuspidal non-Q-critical points are very Zariski-dense in $\mathscr{I}$.

Proof. Let $\mathscr{I}_{\mathrm{ncs}} := \mathscr{I} \cap w^{-1}(\Omega_{\mathrm{ncs}})$. By [19, Prop. 5.15] (and its proof), $\mathscr{I}_{\mathrm{ncs}}$ is very Zariski-dense in $\mathscr{I}$ and every $y \in \mathscr{I}_{\mathrm{ncs}}$ is classical cuspidal non-Q-critical, from which the result follows. □

In the proof of Theorem 7.6(c), we need a lemma. Note $w^{-1}(\lambda) \cap \mathscr{C}$ is a finite set for all $\lambda \in \Omega$.

Lemma 7.15. *Let $\mathscr{C}$ be as in §7.3 and $\lambda \in \Omega_{\mathrm{ncs}}$. Reduction modulo $\mathfrak{m}_\lambda$ induces an isomorphism*

$$\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h} \otimes_{\mathbf{T}_{\Omega,h}(K(\tilde{\pi}))} \mathbf{T}_{\Omega,\mathscr{C}}/\mathfrak{m}_\lambda \cong \bigoplus_{y \in w^{-1}(\lambda)\cap\mathscr{C}} \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\lambda)_{\mathfrak{m}_y}.$$

Proof. This is local at $\lambda \in \mathscr{W}^Q_{\lambda_\pi}$, so we are free to shrink Ω to a neighbourhood of λ with

$$\mathscr{C} = \textstyle\bigsqcup_{y \in w^{-1}(\lambda)\cap\mathscr{C}} \mathscr{C}_y,$$

with each $\mathscr{C}_y = \mathrm{Sp}(\mathbf{T}_y)$ connected affinoid, and with $w^{-1}(\lambda) \cap \mathscr{C}_y = \{y\}$. Note that $\mathscr{C}$ itself can be disconnected over this smaller Ω. As

$$\mathbf{T}_{\Omega,\mathscr{C}} = \oplus_{y \in w^{-1}(\lambda)\cap\mathscr{C}} \mathbf{T}_y,$$

we have

$$\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h} \otimes_{\mathbf{T}_{\Omega,h}(K(\tilde{\pi}))} \mathbf{T}_{\Omega,\mathscr{C}}/\mathfrak{m}_\lambda \cong \bigoplus_{y \in w^{-1}(\lambda)\cap\mathscr{C}} \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h} \otimes_{\mathbf{T}_{\Omega,h}(K(\tilde{\pi}))} \mathbf{T}_y/\mathfrak{m}_\lambda.$$

As y is the unique point of $\mathscr{C}_y$ above λ, in each summand of the right-hand side, reduction mod $\mathfrak{m}_\lambda$ factors through localisation at $\mathfrak{m}_y$; and since $\mathscr{C}_y$ is the connected component through y, each

$$\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h}\otimes_{\mathbf{T}_{\Omega,h}}\mathbf{T}_y\subset \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)$$

is a summand, and

$$[\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h}\otimes_{\mathbf{T}_{\Omega,h}}\mathbf{T}_y]_{\mathfrak{m}_y}=\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)_{\mathfrak{m}_y}.$$

Thus

$$\begin{aligned}\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h}\otimes_{\mathbf{T}_{\Omega,h}(K(\tilde\pi))}\mathbf{T}_y/\mathfrak{m}_\lambda &\cong \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)_{\mathfrak{m}_y}\otimes_\Lambda \Lambda/\mathfrak{m}_\lambda\\ &\cong \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\lambda)_{\mathfrak{m}_y},\end{aligned}$$

where the last isomorphism is Proposition 7.8, as each such y has non-Q-critical slope (since $\lambda\in\Omega_{\mathrm{ncs}}$). Combining the last two displayed equations gives the Lemma. □

We now complete the proof of Theorem 7.6(c). Recall $\mathscr{C}$ is the connected component of $\mathscr{E}_{\Omega,h}(K(\tilde\pi))$ containing $x_{\tilde\pi}$, and let $\mathscr{C}_{\mathrm{nc}}$ denote the set of classical cuspidal non-Q-critical points $y\in\mathscr{C}$. Let $\mathscr{C}^{\mathrm{Sha}}_{\mathrm{nc}}$ be the subset of points $y\in\mathscr{C}_{\mathrm{nc}}$ such that π_y admits a global $(\eta_0|\cdot|^{\mathsf{w}_y},\psi)$-Shalika model, where w_y is the purity weight of $\lambda_y=w(y)$. Theorem 7.6(c) then follows from:

Proposition 7.16. *Suppose the hypotheses of Theorem 7.6(c). Up to shrinking Ω, there is an irreducible component $\mathscr{I}\subset\mathscr{C}\subset\mathscr{E}_{\Omega,h}(K(\tilde\pi))$ such that:*

- *$\mathscr{I}$ contains $x_{\tilde\pi}(K(\tilde\pi))$,*
- *$\dim(\mathscr{I})=\dim(\Omega)$, and*
- *$\mathscr{I}\cap\mathscr{C}^{\mathrm{Sha}}_{\mathrm{nc}}$ is very Zariski-dense in $\mathscr{I}$.*

Proof. Let $\mathscr{C}^{\mathrm{Sha}}$ be the Zariski-closure of $\mathscr{C}^{\mathrm{Sha}}_{\mathrm{nc}}$ in $\mathscr{C}$. We claim:

Claim 7.17. *$w(\mathscr{C}^{\mathrm{Sha}}_{\mathrm{nc}})$ is a (very) Zariski-dense subset of Ω.*

The claim implies that $w(\mathscr{C}^{\mathrm{Sha}})=\Omega$, that is $\mathscr{C}^{\mathrm{Sha}}$ has full support in Ω. Given this, we conclude that $\mathscr{C}^{\mathrm{Sha}}$ has an irreducible component $\mathscr{I}$ of dimension $\dim(\Omega)$, as $\mathscr{E}_{\Omega,h}(K(\tilde\pi))$ has finitely many irreducible components (see Corollary 7.11). Then $\mathscr{I}$ satisfies the conditions we require. Thus the proposition follows from the claim.

Proof of claim. Fix a character χ of conductor p^β and $j\in\mathrm{Crit}(\lambda_\pi)$ such that $L^{(p)}(\pi\otimes\chi,j+\frac12)\neq0$ (which exist by hypothesis). Considering $\chi\chi^j_{\mathrm{cyc}}\in\mathcal{A}(\mathrm{Gal}_p,\mathcal{O}_\Omega)$ via the structure map $L\to\mathcal{O}_\Omega$, define

$$\begin{aligned}\mathrm{Ev}^\Omega_{\chi,j}:\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)&\longrightarrow\mathcal{O}_\Omega\\ \Phi\qquad&\longmapsto\int_{\mathrm{Gal}_p}\chi\chi^j_{\mathrm{cyc}}\cdot\mathrm{Ev}^{\eta_0}_\beta(\Phi),\end{aligned}$$

for $\mathrm{Ev}^{\eta_0}_\beta$ as in (6.15). Restricting under (7.7), $\mathrm{Ev}^\Omega_{\chi,j}$ defines a map $\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h}\otimes_{\mathbf{T}_{\Omega,h}}\mathbf{T}_{\Omega,\mathscr{C}}\to\mathcal{O}_\Omega$, which we can evaluate at the class $\Phi_{\mathscr{C}}$ from (7.8). By construction $\mathrm{sp}_{\lambda_\pi}(\Phi_{\mathscr{C}})=\Phi_{\tilde\pi}$, so

$$\mathrm{sp}_{\lambda_\pi}\circ\mathrm{Ev}^{\eta_0}_\beta(\Phi_{\mathscr{C}})=\mathrm{Ev}^{\eta_0}_\beta(\Phi_{\tilde\pi})$$

by Proposition 6.15. Thus

$$\begin{aligned}\big[\mathrm{Ev}^{\Omega}_{\chi,j}(\Phi_{\mathscr{C}})\big](\lambda_\pi) &= \int_{\mathrm{Gal}_p} \chi\chi_{\mathrm{cyc}}^{j}\cdot \mathrm{sp}_\lambda\left(\mathrm{Ev}^{\eta_0}_{\beta}(\Phi_{\mathscr{C}})\right)\\ &= \int_{\mathrm{Gal}_p} \chi\chi_{\mathrm{cyc}}^{j}\cdot \mathrm{Ev}^{\eta_0}_{\beta}(\Phi_{\tilde\pi}) \neq 0,\end{aligned}$$

where non-vanishing follows as in the proof of Corollary 7.12. As the non-vanishing locus is open, up to shrinking Ω we may assume that $\mathrm{Ev}^{\Omega}_{\chi,j}(\Phi^{\epsilon}_{\mathscr{C}}) \in \mathcal{O}_\Omega$ is everywhere non-vanishing.

Now let λ be any weight in Ω_{ncs}, the set from §7.4. Since $\mathscr{E}_{\Omega,h}(K(\tilde\pi))$ is finite over Ω, the preimage $w^{-1}(\lambda)\cap\mathscr{C}$ is a finite set. From Lemma 7.15, for $\lambda\in\Omega_{\mathrm{ncs}}$ we may write

$$\mathrm{sp}_\lambda(\Phi_{\mathscr{C}}) = \oplus_y \Phi_y, \tag{7.10}$$

with each $\Phi_y \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\lambda)_{\mathfrak{m}_y}$. Recalling r_y from (3.10), for each such y, we have

$$r_\lambda(\Phi_y) = \oplus_\epsilon r_\lambda(\Phi_y^\epsilon) \in \bigoplus_\epsilon \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{V}_\lambda^\vee)^\epsilon_{\mathfrak{m}_y} = \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{V}_\lambda^\vee)_{\mathfrak{m}_y},$$

projecting into the decomposition over ϵ of §2.3.4. Now by combining Proposition 6.15 and Lemma 6.18, we have a commutative diagram

$$\begin{array}{ccc}\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^\epsilon & \xrightarrow{\ \mathrm{Ev}^{\Omega}_{\chi,j}\ } & \mathcal{O}_\Omega\\ \Big\downarrow{\scriptstyle r_\lambda\circ\mathrm{sp}_\lambda} & & \Big\downarrow{\scriptstyle \mathrm{sp}_\lambda}\\ \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{V}_\lambda^\vee)^\epsilon & \xrightarrow{\ \mathcal{E}^{j,\eta_0}_{\chi}\ } & L.\end{array}$$

Combining this with (7.10), and the fact that $\mathrm{Ev}^{\Omega}_{\chi,j}(\Phi_{\mathscr{C}})$ is everywhere non-vanishing, we deduce

$$[\mathrm{Ev}^{\Omega}_{\chi,j}(\Phi_{\mathscr{C}})](\lambda) = \sum_{y\in w^{-1}(\lambda)\cap\mathscr{C}}\ \sum_\epsilon \mathcal{E}^{j,\eta_0}_{\chi}\left(r_\lambda(\Phi_y^\epsilon)\right) \neq 0.$$

Hence at least one of the terms in the sum is non-zero. By Proposition 5.15, we deduce that if this term corresponds to the point y, then π_y admits an $(\eta_0|\cdot|^{\mathsf{w}},\psi)$-Shalika model (where w is the purity weight of λ). Thus above each $\lambda\in\Omega_{\mathrm{ncs}}$, there exists at least one classical point $y\in\mathscr{C}$ corresponding to an automorphic representation π_y admitting a Shalika model. In particular, we deduce that $\Omega_{\mathrm{ncs}}\subset w(\mathscr{C}^{\mathrm{Sha}}_{\mathrm{nc}})$. Thus $w(\mathscr{C}^{\mathrm{Sha}}_{\mathrm{nc}})$ is very Zariski-dense in Ω, as required. □

7.5. Proof of Thm. 7.6(d–e): Étaleness of Shalika families. At this point we perform a delicate switch in level to prove Theorem 7.6(d–e). Fix $\epsilon\in\{\pm1\}^\Sigma$. A key fact about the level $K_1(\tilde\pi)$ from (7.2) is the following:

Proposition 7.18. *The vector space* $\mathrm{H}^t_{\mathrm{c}}(S_{K_1(\tilde\pi)},\mathscr{V}^\vee_{\lambda_\pi})^\epsilon_{\mathfrak{m}_{\tilde\pi}}$ *is 1-dimensional.*

Proof. The $\mathfrak{m}_{\tilde\pi}$-torsion in $\pi_f^{K_1(\tilde\pi)}$ is a line; locally, this follows for $v\nmid p$ by (7.1), and for $\mathfrak{p}|p$ by (2.18). By Proposition 2.3,

$$\dim_{\overline{\mathbf{Q}}_p}\ \mathrm{H}^t_{\mathrm{c}}(S_{K_1(\tilde\pi)},\mathscr{V}^\vee_{\lambda_\pi}(\overline{\mathbf{Q}}_p))^\epsilon_{\mathfrak{m}_{\tilde\pi}} = 1.$$

We descend to L via §2.10. □

Recall $\Lambda=\mathcal{O}_{\Omega,\lambda_\pi}$ is the algebraic localisation. Taking ϵ-parts of Proposition 7.8 gives isomorphisms

$$\begin{aligned}\mathrm{H}^t_{\mathrm{c}}(S_{K_1(\tilde\pi)},\mathscr{D}_\Omega)^\epsilon_{\mathfrak{m}_{\tilde\pi}}\otimes_\Lambda \Lambda/\mathfrak{m}_{\lambda_\pi} &\cong \mathrm{H}^t_{\mathrm{c}}(S_{K_1(\tilde\pi)},\mathscr{D}_{\lambda_\pi})^\epsilon_{\mathfrak{m}_{\tilde\pi}}\\ &\cong \mathrm{H}^t_{\mathrm{c}}(S_{K_1(\tilde\pi)},\mathscr{V}^\vee_{\lambda_\pi})^\epsilon_{\mathfrak{m}_{\tilde\pi}}\end{aligned}\tag{7.11}$$

of 1-dimensional vector spaces (where the second isomorphism is non-Q-criticality). In particular, there exists a point $x_{\tilde{\pi}}^{\epsilon}(K_1(\tilde{\pi}))$ in $\mathscr{E}_{\Omega,h}^{\epsilon}(K_1(\tilde{\pi}))$ corresponding to $\tilde{\pi}$. Let

$$\mathbf{T}_{\Omega,\tilde{\pi}}^{\epsilon}(K_1(\tilde{\pi})) = \mathbf{T}_{\Omega,h}^{\epsilon}(K_1(\tilde{\pi}))_{\mathfrak{m}_{\tilde{\pi}}},$$

which acts on $\mathrm{H}_{\mathrm{c}}^t(S_{K_1(\tilde{\pi})}, \mathscr{D}_\Omega)_{\mathfrak{m}_{\tilde{\pi}}}^{\leqslant h,\epsilon}$ (see Definition 2.9).

Proposition 7.19. *There exists a proper ideal $I_{\tilde{\pi}}^{\epsilon} \subset \Lambda$ such that*

$$\mathbf{T}_{\Omega,\tilde{\pi}}^{\epsilon}(K_1(\tilde{\pi})) \cong \Lambda/I_{\tilde{\pi}}^{\epsilon}.$$

Proof. Since $\mathrm{H}_{\mathrm{c}}^t(S_{K_1(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h}$ is a finitely generated $\mathcal{O}_\Omega$-module, $\mathrm{H}_{\mathrm{c}}^t(S_{K_1(\tilde{\pi})}, \mathscr{D}_\Omega)_{\mathfrak{m}_{\tilde{\pi}}}^{\epsilon}$ is finitely generated over Λ; then Nakayama's lemma applied to (7.11) implies that $\mathrm{H}_{\mathrm{c}}^t(S_{K_1(\tilde{\pi})}, \mathscr{D}_\Omega)_{\mathfrak{m}_{\tilde{\pi}}}^{\epsilon}$ is non-zero and generated by a single element over Λ. In particular, it is isomorphic to $\Lambda/I_{\tilde{\pi}}^{\epsilon}$ for some proper ideal $I_{\tilde{\pi}}^{\epsilon} \subset \Lambda$. Now, we know that $\mathbf{T}_{\Omega,\tilde{\pi}}^{\epsilon}(K_1(\tilde{\pi}))$ is the image of the Hecke algebra in

$$\mathrm{End}_\Lambda\left(\mathrm{H}_{\mathrm{c}}^t(S_{K_1(\tilde{\pi})}, \mathscr{D}_\Omega)_{\mathfrak{m}_{\tilde{\pi}}}^{\epsilon}\right) \cong \mathrm{End}_\Lambda(\Lambda/I_{\tilde{\pi}}^{\epsilon}) \cong \Lambda/I_{\tilde{\pi}}^{\epsilon}.$$

But this image contains 1, so $\mathbf{T}_{\Omega,\tilde{\pi}}^{\epsilon}(K_1(\tilde{\pi}))$ must be everything, giving the result. □

To prove (d) and (e), we will combine Proposition 7.19 with Corollary 7.11(i) (which implies $\mathbf{T}_{\Omega,\tilde{\pi}}(K(\tilde{\pi}))$ is Λ-torsion free). To switch between levels $K(\tilde{\pi})$ and $K_1(\tilde{\pi})$, recall the connected component $\mathscr{C}(K(\tilde{\pi}))$ from §7.3, and:

- Let $\mathscr{C}_{\mathrm{nc}}^{\mathrm{lgc}}(K(\tilde{\pi}))$ be the set of classical cuspidal non-Q-critical points $y \in \mathscr{C}(K(\tilde{\pi}))$ such that Local-Global Compatibility holds for π_y at all v. Note (as explained in §7.1.2) that $\mathscr{C}_{\mathrm{nc}}^{\mathrm{Sha}}(K(\tilde{\pi})) \subset \mathscr{C}_{\mathrm{nc}}^{\mathrm{lgc}}(K(\tilde{\pi}))$.
- Let $\mathscr{C}^{\mathrm{lgc}}(K(\tilde{\pi}))$ be the Zariski-closure of $\mathscr{C}_{\mathrm{nc}}^{\mathrm{lgc}}(K(\tilde{\pi}))$, equipped with the induced reduced rigid analytic structure. This contains (the nilreductions of) all Shalika families through $x_{\tilde{\pi}}(K(\tilde{\pi}))$, so by Proposition 7.16, it contains an irreducible component of dimension $\dim(\Omega)$.

In the next subsection, we prove:

Proposition 7.20. *Let $\tilde{\pi}$ satisfy the hypotheses of Theorem 7.6(a–c), and suppose ρ_π is irreducible. Then, up to shrinking Ω, for all $y \in \mathscr{C}_{\mathrm{nc}}^{\mathrm{lgc}}(K(\tilde{\pi}))$ and for all $v \in S$, the Whittaker conductors of π_v and $\pi_{y,v}$ are equal. In particular, $\pi_{y,f}^{K_1(\tilde{\pi})} \neq 0$.*

Corollary 7.21. *For any $\epsilon \in \{\pm 1\}^\Sigma$ there exists a closed immersion*

$$\iota : \mathscr{C}^{\mathrm{lgc}}(K(\tilde{\pi})) \hookrightarrow \mathscr{E}_{\Omega,h}^{\epsilon}(K_1(\tilde{\pi}))$$

sending $x_{\tilde{\pi}}(K(\tilde{\pi}))$ to $x_{\tilde{\pi}}^{\epsilon}(K_1(\tilde{\pi}))$.

Proof. This is a straightforward application of [55, Thm. 3.2.1], with the same Hecke algebra $\mathcal{H}$ and weight space Ω (by Remark 3.5) on both sides, with the identity maps between them. To apply this, it suffices to prove that we have this transfer on a Zariski-dense set of points. The subset of $y \in \mathscr{C}_{\mathrm{nc}}^{\mathrm{lgc}}(K(\tilde{\pi}))$ that have non-Q-critical slope is Zariski-dense in $\mathscr{C}^{\mathrm{lgc}}(K(\tilde{\pi}))$. For such y, by Proposition 7.20 and (2.11), we know $\mathrm{H}_{\mathrm{c}}^t(S_{K_1(\tilde{\pi})}, \mathscr{V}_\lambda^\vee)_{\mathfrak{m}_y}^{\epsilon} \neq 0$; then by (7.11) (cf. Proposition 7.8, Remark 7.9) there is a point $y(K_1(\tilde{\pi})) \in \mathscr{E}_{\Omega,h}^{\epsilon}(K_1(\tilde{\pi}))$ attached to the same Hecke eigensystem as y. The transfer is then $y \mapsto y(K_1(\tilde{\pi}))$ on the (Zariski-dense) subset of non-Q-critical slope y. □

Corollary 7.22. *Let $\tilde{\pi}$ satisfy the hypotheses of Theorem 7.6(a–d), and let $\epsilon \in \{\pm 1\}^\Sigma$. Then* *(i) The weight map* $\mathscr{E}^{\epsilon}_{\Omega,h}(K_1(\tilde{\pi})) \to \Omega$ *is étale at* $x_{\tilde{\pi}}(K_1(\tilde{\pi}))$.

(ii) The natural map $\mathscr{E}^{\epsilon}_{\Omega,h}(K_1(\tilde{\pi})) \hookrightarrow \mathscr{E}_{\Omega,h}(K_1(\tilde{\pi}))$ *is locally an isomorphism at* $x_{\tilde{\pi}}(K_1(\tilde{\pi}))$.

(iii) The weight map $\mathscr{E}_{\Omega,h}(K_1(\tilde{\pi})) \to \Omega$ *is étale at* $x_{\tilde{\pi}}(K_1(\tilde{\pi}))$.

Proof. (i) It suffices to prove that the ideal $I^{\epsilon}_{\tilde{\pi}}$ from Proposition 7.19 is zero. Suppose it is not; then every irreducible component of $\mathscr{E}^{\epsilon}_{\Omega,h}(K_1(\tilde{\pi}))$ through $x^{\epsilon}_{\tilde{\pi}}(K_1(\tilde{\pi}))$ has dimension less than $\dim(\Omega)$. But $\mathscr{C}^{\mathrm{lgc}}(K(\tilde{\pi}))$ has a component of dimension $\dim(\Omega)$ through $x^{\epsilon}_{\tilde{\pi}}(K(\tilde{\pi}))$ by the discussion before Proposition 7.20; under ι this maps to a component of dimension $\dim(\Omega)$, which is a contradiction.

(ii) Let $\epsilon \neq \epsilon'$, and $\mathscr{C}^{\epsilon}, \mathscr{C}^{\epsilon'}$ be the connected components through $\tilde{\pi}$ of $\mathscr{E}^{\epsilon}_{\Omega,h}(K_1(\tilde{\pi}))$ and $\mathscr{E}^{\epsilon'}_{\Omega,h}(K_1(\tilde{\pi}))$ respectively. By above, $\mathscr{C}^{\epsilon}$ and $\mathscr{C}^{\epsilon'}$ are étale over Ω and contain Zariski-dense sets $\mathscr{C}^{\epsilon}_{\mathrm{nc}}, \mathscr{C}^{\epsilon'}_{\mathrm{nc}}$ of points corresponding to the same set of Q-refined RACARs $\{\tilde{\pi}_y\}_y$. By another application of [55, Thm. 3.2.1], there exist closed immersions

$$\mathscr{C}^{\epsilon} \hookrightarrow \mathscr{C}^{\epsilon'} \hookrightarrow \mathscr{C}^{\epsilon}$$

over Ω that are the identity on $\{\tilde{\pi}_y\}_y$; hence $\mathscr{C}^{\epsilon}$ and $\mathscr{C}^{\epsilon'}$ are canonically identified, and $\mathscr{C}^{\epsilon}$ is independent of ϵ. At $\tilde{\pi}$, since the Hecke algebra preserves ϵ-parts in cohomology, this means that

$$\mathbf{T}_{\Omega,\tilde{\pi}}(K_1(\tilde{\pi})) = \mathbf{T}^{\epsilon}_{\Omega,\tilde{\pi}}(K_1(\tilde{\pi}))$$

as Λ-modules, and part (ii) follows. Part (iii) is immediate from (i) and (ii). □

Modulo Proposition 7.20, this proves Theorem 7.6(d1). For (d2), let $\mathscr{C}^{\mathrm{Sha}}(K(\tilde{\pi}))^{\mathrm{red}}$ be the nilreduction of $\mathscr{C}^{\mathrm{Sha}}(K(\tilde{\pi}))$. By the discussion before Proposition 7.20, and Corollary 7.21, we have a diagram

$$\begin{array}{ccccc} \mathscr{C}^{\mathrm{Sha}}(K(\tilde{\pi}))^{\mathrm{red}} & \subset & \mathscr{C}^{\mathrm{lgc}}(K(\tilde{\pi})) & \overset{\iota}{\hookrightarrow} & \mathscr{E}^{\epsilon}_{\Omega,h}(K_1(\tilde{\pi})) \\ & \searrow & \downarrow & \swarrow & \\ & & \Omega & & \end{array}$$

As $\mathscr{C}^{\mathrm{Sha}}(K(\tilde{\pi}))^{\mathrm{red}}$ contains an irreducible component of dimension $\dim(\Omega)$, and $\mathscr{E}^{\epsilon}_{\Omega,h}(K_1(\tilde{\pi}))$ is étale over Ω, we deduce $\mathscr{C}^{\mathrm{Sha}}(K(\tilde{\pi}))^{\mathrm{red}}$ is étale over Ω; hence $\mathscr{C}^{\mathrm{Sha}}(K(\tilde{\pi}))$ contains a unique irreducible component, giving (d2). If Local-Global Compatibility holds for all RACARs, then $\mathscr{C}^{\mathrm{lgc}}(K(\tilde{\pi})) = \mathscr{C}(K(\tilde{\pi}))^{\mathrm{red}}$, and the same argument shows this is étale over Ω, giving (e).

7.6. Level-switching: local constancy of conductors. It remains to prove Proposition 7.20. We use Galois theory. Let $y \in \mathscr{C}^{\mathrm{lgc}}_{\mathrm{nc}}(K(\tilde{\pi}))$, with attached p-adic Galois representation

$$\rho_{\pi_y} : \mathrm{G}_F \to \mathrm{GL}_{2n}(L) \subset \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p),$$

depending on $\iota_p : \mathbf{C} \cong \overline{\mathbf{Q}}_p$. Attached to π_y and $v \in S$, we have

- the Whittaker conductor $m(\pi_{y,v})$ from (7.1), and
- the *Artin conductor* $a(\rho_{\pi_y}|_{\mathrm{G}_{F_v}})$ of the restriction of ρ_{π_y} to G_{F_v}, defined by Serre in [74].

Proposition 7.23. *If* $y \in \mathscr{C}^{\mathrm{lgc}}_{\mathrm{nc}}(K(\tilde{\pi}))$, *then for any* $v \nmid p$, *we have* $m(\pi_{y,v}) = a(\rho_{\pi_y}|_{\mathrm{G}_{F_v}})$.

Proof. Let m and a denote the conductors. Let $\rho_{y,v} = \rho_{\pi_y}|_{\mathrm{G}_{F_v}}$, and $\mathrm{WD}(\rho_{y,v})$ its associated Weil–Deligne representation. By Local-Global Compatibility (see §7.1.2) we have

$$\mathrm{WD}(\rho_{y,v})^{\mathrm{F-ss}} = \iota_p \mathrm{rec}_{F_v}(\pi_{y,v} \otimes |\cdot|^{(1-n)/2}).$$

Fix an unramified non-trivial additive character ψ_v of F_v and let $q_v = \#\mathcal{O}_v/\varpi_v$. Then:

– $\rho_{y,v}$ and $\mathrm{WD}(\rho_{y,v})^{\mathrm{F-ss}}$ have the same Artin conductor a (e.g. [80, §8]);
– the map rec_{F_v} preserves ε-factors [48], so $\varepsilon(s, \pi_{y,v}|\cdot|^{(1-n)/2}, \psi_v) = \varepsilon(s, \mathrm{WD}(\rho_{y,v}), \psi_v)$;
– by [78, (3.4.5)], we have $\varepsilon(s, \mathrm{WD}(\rho_{y,v}), \psi_v) = C \cdot q_v^{-as}$ for $C \in \mathbf{C}^\times$ independent of s;
– by [51, (1),Thm. §5], $\varepsilon(s, \pi_{y,v}|\cdot|^{(1-n)/2}, \psi_v) = \varepsilon(s + (1-n)/2, \pi_{y,v}, \psi_v) = C' \cdot q_v^{-m(1-n)/2} \cdot q_v^{-ms}$, for $C' \in \mathbf{C}^\times$ independent of s.

Hence $C = C' \cdot q_v^{-m(1-n)/2}$ and $a = m$, as required. □

The study of $m(\pi_{y,v})$ in families is thus reduced to that of $a(\rho_{\pi_y}|_{\mathrm{G}_{F_v}})$, and hence can be studied via Galois theory. For simplicity, let $\mathscr{C}^{\mathrm{lgc}} = \mathscr{C}^{\mathrm{lgc}}(K(\tilde{\pi}))$ and $\mathscr{C}^{\mathrm{lgc}}_{\mathrm{nc}} = \mathscr{C}^{\mathrm{lgc}}_{\mathrm{nc}}(K(\tilde{\pi}))$.

Lemma 7.24. *Suppose ρ_π is irreducible. Then possibly shrinking Ω, there exists a Galois representation*

$$\rho_{\mathscr{C}^{\mathrm{lgc}}} : \mathrm{G}_F \to \mathrm{GL}_2(\mathcal{O}_{\mathscr{C}^{\mathrm{lgc}}})$$

such that for all $y \in \mathscr{C}^{\mathrm{lgc}}_{\mathrm{nc}}$, we have $\rho_{\pi_y} = \rho_{\mathscr{C}^{\mathrm{lgc}}} \pmod{\mathfrak{m}_y}$.

Proof. Let $\nu = (1, 0, ..., 0) \in X_*^+(T_{2n})$. For each $v \notin S \cup \{\mathfrak{p}|p\}$, we have a Hecke operator $T_{\nu,v} \in \mathcal{H}$ as in §2.4. If $y \in \mathscr{C}^{\mathrm{lgc}}_{\mathrm{nc}}$ corresponds to the character $\Psi_y : \mathcal{H} \otimes L \to L$, then we have

$$\Psi_y(T_{\nu,v}) = \mathrm{Tr}(\rho_{\pi_y}(\mathrm{Frob}_v))$$

(see e.g. [32, Cor. 7.3.4]). In particular, property (H) of [32, §7.1] holds, where we take a_v *ibid.* to be the image of $T_{\nu,v}$ in $\mathcal{O}_{\mathscr{C}^{\mathrm{lgc}}}$ under the natural map. Then by [32, Lem. 7.1.1], there exists a $2n$-dimensional Galois pseudo-character

$$t_{\mathscr{C}^{\mathrm{lgc}}} : \mathrm{G}_F \to \mathcal{O}_{\mathscr{C}^{\mathrm{lgc}}}$$

over $\mathscr{C}^{\mathrm{lgc}}$ such that for all $v \notin S \cup \{\mathfrak{p}|p\}$ and all $y \in \mathscr{C}^{\mathrm{lgc}}_{\mathrm{nc}}$, we have

$$\mathrm{sp}_y(t_{\mathscr{C}^{\mathrm{lgc}}}(\mathrm{Frob}_v)) = \rho_{\pi_y}(\mathrm{Frob}_v).$$

As ρ_π is irreducible, by [22, Lem. 4.3.7], there exists a lift of $t_{\mathscr{C}^{\mathrm{lgc}}}$ to a Galois representation

$$\rho_{\mathscr{C}^{\mathrm{lgc}}} : \mathrm{G}_F \to \mathrm{GL}_{2n}(\mathcal{O}_{\mathscr{C}^{\mathrm{lgc}}})$$

with $t_{\mathscr{C}^{\mathrm{lgc}}} = \mathrm{tr}(\rho_{\mathscr{C}})$; and $\rho_{\pi_y} = \rho_{\mathscr{C}^{\mathrm{lgc}}} \pmod{\mathfrak{m}_y}$. □

Proposition 7.25. *Let $v \in S$ with residue characteristic $\ell \neq p$. After possibly shrinking Ω, the Artin conductor $a(\rho_{\pi_y}|_{\mathrm{G}_{F_v}})$ is constant as y varies in $\mathscr{C}^{\mathrm{lgc}}$. Hence Proposition 7.20 holds.*

Proof. Let $(r, N) = \mathrm{WD}(\rho_{\mathscr{C}^{\mathrm{lgc}}}|_{\mathrm{G}_{F_v}})$ be the family of Weil–Deligne representations associated to $\rho_{\mathscr{C}^{\mathrm{lgc}}}$ at v [22, Lem. 7.8.14]. By construction, the specialisation (r_y, N_y) of (r, N) at $y \in \mathscr{C}^{\mathrm{lgc}}$ is the Weil–Deligne representation attached to $\rho_{\pi_y}|_{\mathrm{G}_{F_v}}$, and then by definition (see [80, §7]), we have

$$a(\rho_{\pi_y}|_{\mathrm{G}_{F_v}}) = a(r_y) + \dim\left(r_y^{\mathrm{I}_v}\right) - \dim[\ker(N_y) \cap r_y^{\mathrm{I}_v}], \tag{7.12}$$

with $a(r_y)$ the conductor of r_y (depending only on $r_y|_{\mathrm{I}_v}$). By [22, Lem. 7.8.17], $r|_{\mathrm{I}_v}$ is locally constant over $\mathscr{C}^{\mathrm{lgc}}$, so we can shrink Ω so $a(r_y)$ and $\dim(r_y^{\mathrm{I}_v})$ are constant as y varies in $\mathscr{C}^{\mathrm{lgc}}$.

Now note that since π is essentially self-dual, the specialisation $(r_x, N_x) = \mathrm{WD}(\rho_\pi|_{\mathrm{G}_{F_v}})$ is pure. Indeed, it suffices to check this after passing to the base-change Π of π to a quadratic CM extension F'/F in which v splits as $w\overline{w}$. By [34, Lem. 4.1.4, §4.3] there exists an algebraic Hecke character χ over F' such that $\Pi' := \Pi \otimes \chi$ is self-dual, and then [29, Thm. 1.2] shows that Π'_w is tempered, so has pure Weil–Deligne representation. But purity is preserved by algebraic twist.

Combining [22, Prop. 7.8.19] with [71, Thm. 3.1(2)], purity at x implies that for all y in a neighbourhood of x, we have $N_x \sim N_y$ in the sense of [22, Defs. 6.5.1, 7.8.2]. This implies that $\dim[\ker(N_y) \cap r_y^{\mathrm{I}_v}]$ – and hence $a(\rho_{\pi_y}|_{\mathrm{G}_{F_v}})$, by (7.12) – is constant for y in a neighbourhood of x.

Proposition 7.20 now follows by combining this with Proposition 7.23. □

7.7. Remarks on symplectic components. The space $\mathscr{E}_{\Omega,h}(K)$ studied in this section is a local piece of a global parabolic eigenvariety $\mathscr{E}_{\lambda_0}^{Q}(K)$ varying over $\mathscr{W}_{\lambda_0}^{Q}$, constructed in [19, §5.2]. (Precisely, we take $* = t$ in the notation *op. cit.*; that is, this is a 'top degree' eigenvariety). Here λ_0 is any algebraic weight in Ω. We have described its local geometry at certain Shalika points. We now comment on global implications, proving:

Theorem 7.26. *Let $\mathscr{I} \subset \mathscr{E}_{\lambda_0}^{Q}(K)$ be an irreducible component, where K is some parahoric-at-p level. Suppose $\mathscr{I}$ contains a Shalika point $x_{\tilde\pi}$ attached to a Q-refined RASCAR $\tilde\pi$ that is spherical and regular at p and satisfies the hypotheses (a–d) of Theorem 7.6, and that $K = K_1(\tilde\pi)$. Then every classical point of $\mathscr{I}$ with non-Q-critical slope and regular weight is a Shalika point.*

We will prove (in Theorem 7.34) a stronger result. Let $\mathcal{G} := \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GSpin}_{2n+1}$ be the split spin group. If $\tilde\pi_y$ is a Shalika point in $\mathscr{I}$, then π_y is the functorial transfer of a RACAR Π_y of $\mathcal{G}(\mathbf{A})$ (see §1.1). By [14, §3.1], there is a refinement $\tilde\Pi_y$ of Π_y corresponding to $\tilde\pi_y$. We show there is an irreducible component $\mathscr{I}^{\mathcal{G}}$ in a parabolic eigenvariety for $\mathcal{G}$, and rigid analytic maps between the (nilreductions) of $\mathscr{I}$ and $\mathscr{I}^{\mathcal{G}}$ that interpolate the correspondence $\tilde\pi_y \leftrightarrow \tilde\Pi_y$ and induce bijections on their sets of points. Thus *every* eigensystem in $\mathscr{I}$, classical or not, is *symplectic*, a functorial transfer from $\mathcal{G}$. For non-Q-critical slope classical points of regular weight, symplectic is equivalent to Shalika (see Proposition 7.33), so the theorem follows.

The proof occupies the rest of this section; we sketch it now. One has a natural map from the Hecke algebra for G to that for $\mathcal{G}$, compatible with Langlands functoriality, induced by a map $\jmath^\vee$ on cocharacters. It also admits a natural section $\iota^\vee$. Using $\iota^\vee$, and properties of Langlands functoriality, one can transfer a Zariski-dense set of Shalika points in $\mathscr{I}$ from the eigenvariety for G to that for $\mathcal{G}$. Using an idea of Chenevier, this interpolates to a map f on the nilreduction of $\mathscr{I}$. Let $\mathscr{I}^{\mathcal{G}}$ be the irreducible component containing the (irreducible) image. Applying the same argument in reverse, with $\jmath^\vee$, gives a map g the other way inverse to f on points.

Remark 7.27. A more detailed study of these phenomena, for all parahoric levels, is the subject of [14]. As a flavour: in the Iwahori-level eigenvariety, the analogue of Theorem 7.26 (for classical points) should hold, but the stronger analogue (on non-classical points) should not. In the language *op. cit.*, take a non-critical slope Iwahori refinement of π that is optimally Q-spin. This varies in a $dn+1$-dimensional component $\mathscr{I}$ in the Iwahori eigenvariety, but the symplectic locus is a closed $d+1$-dimensional subspace. In [14] we conjecture that the classical points in $\mathscr{I}$ lie in the symplectic subspace; but there should exist non-classical non-symplectic points in $\mathscr{I}$.

7.7.1. Hecke algebras for G and $\mathcal{G}$. Fix a Borel pair $(\mathcal{B}, \mathcal{T})$ in $\mathcal{G}$, as in [14, §2]. Attached to $Q \subset G$ is a parabolic $\mathcal{Q} \subset \mathcal{G}$, described in [14, §2.1]. Let $\mathcal{J}_{\mathfrak{p}}$ be the associated parahoric subgroup. Let $\mathcal{U}_{\mathfrak{p}}^\circ$ be the associated normalised Hecke operator ($\mathcal{U}_{\mathfrak{p},n}^\circ$ in the notation *op. cit.*).

Let S be the set of finite primes $v \nmid p$ where K_v is not maximal hyperspecial, and let $\mathcal{K} = \prod_v \mathcal{K}_v \subset \mathcal{G}(\mathbf{A}_{F,f})$ be open compact such that $\mathcal{K}_v$ is maximal hyperspecial for every $v \notin S \cup \{\mathfrak{p}|p\}$, sufficiently small at $v \in S$, and $\mathcal{K}_{\mathfrak{p}} = \mathcal{J}_{\mathfrak{p}}$. Let $(\mathcal{H}^{\mathcal{G}})' = \mathbf{Q}_p[\mathcal{T}_{\nu,v} : \nu \in X_*^+(\mathcal{T}), v \notin S \cup \{\mathfrak{p}|p\}]$ be the spherical Hecke algebra for $\mathcal{K}$, where $X_*^+(\mathcal{T})$ is the space of $\mathcal{B}$-dominant cocharacters of $\mathcal{T}$ and $\mathcal{T}_{\nu,v} = [\mathcal{K}_v \nu(\varpi_v)\mathcal{K}_v]$ (as in Definition 2.1). Let $\mathcal{H}^{\mathcal{G}} = (\mathcal{H}^{\mathcal{G}})'[\mathcal{U}_{\mathfrak{p}}^\circ : \mathfrak{p}|p]$.

Henceforth replace the $\mathbf{Z}$-module $\mathcal{H}$ (from §2.9) with $\mathcal{H} \otimes_{\mathbf{Z}} \mathbf{Q}_p$. In [14, §2], a map $\jmath^\vee : X_*(T) \to X_*(\mathcal{T})$ is defined. This induces a map

$$\jmath^\vee : \mathcal{H} \to \mathcal{H}^{\mathcal{G}}, \qquad U_{\mathfrak{p}}^\circ \mapsto \mathcal{U}_{\mathfrak{p}}^\circ, \ \ T_{\nu,v} \mapsto \mathcal{T}_{\jmath^\vee(\nu),v}.$$

In the other direction, there is a natural 'section'

$$\iota^\vee : X_*(\mathcal{T}) \longrightarrow X_*(T) \otimes_{\mathbf{Z}} \mathbf{Z}[1/n]$$

such that $\iota^\vee \circ \jmath^\vee$ is the identity, given in the notation *op. cit.* by

$$\iota^\vee(f_i^*) = e_i^*, \qquad \iota^\vee(f_0^*) = (e_1^* + \cdots + e_n^*)/n.$$

The denominator means this does not, however, induce a map $\mathcal{H}^{\mathcal{G}} \to \mathcal{H}$. To get around this, for $v \notin S \cup \{\mathfrak{p}|p\}$ let

$$Z_v := T_{e_1^*+\cdots+e_n^*,v} = [K_v \mathrm{diag}(\varpi_v, ..., \varpi_v)K_v], \qquad \mathcal{Z}_v = \mathcal{T}_{f_0^*,v}$$

be the operators attached to $e_1^* + \cdots + e_n^*$ and f_0^* respectively. Then $\jmath^\vee(Z_v) = \mathcal{Z}_v^n$. Any map $\mathcal{H}^{\mathcal{G}} \to \mathcal{H}$ induced by $\iota^\vee$ must send $\mathcal{Z}_v$ to an nth root of Z_v. We now make sense of this.

The operators Z_v and $\mathcal{Z}_v$ act respectively by $\mathrm{diag}(\varpi_v, ..., \varpi_v)$ and $f_0^*(\varpi_v)$, elements of the centre of $G(F_v)$ and $\mathcal{G}(F_v)$ (by [4, Prop. 2.3]). Hence they act by the central character evaluated at ϖ_v. If π is a RASCAR of $G(\mathbf{A})$ with an (η, ψ)-Shalika model, then its central character is η^n, and it is the transfer of a RACAR Π of $\mathcal{G}(\mathbf{A})$ whose central character is η (by [4, p.178]).

This observation allows us to formally define an nth root of Z_v over the irreducible component $\mathscr{I}$. Note (as in Definition 7.2) Z_v acts on cohomology at any (η_0, ψ)-Shalika point y by $\big[\eta_0(\varpi_v)|\varpi_v|^{\mathsf{w}_y}\big]^n$. This varies analytically over any affinoid $\Omega \subset \mathscr{W}_{\lambda_\pi}^Q$; let

$$\eta_\Omega(\varpi_v) := \eta_0(\varpi_v) \cdot \mathsf{w}_\Omega(|\varpi_v|) \in \mathcal{O}_\Omega^\times,$$

for w_Ω as in (3.7). Note this is well-defined as $v \nmid p$, so $|\varpi_v| \in \mathbf{Z}_p^\times$. Then $\eta_\Omega(\varpi_v)^n$ interpolates the action of Z_v on (η_0, ψ)-Shalika points π_y in $\mathscr{I}$ above Ω. Such points are Zariski-dense in $\mathscr{I}$ by Theorem 7.6, so we deduce Z_v acts via the functions $\eta_\Omega(\varpi_v)^n$ over all of $\mathscr{I}$.

Definition 7.28. Let z_v be a formal variable, and let

$$\tilde{\mathcal{H}} := \mathcal{H}\Big[z_v : v \notin S \cup \{\mathfrak{p}|p\}\Big]/(Z_v - z_v^n).$$

We may summarise much of the above discussion via:

Lemma 7.29. *The map $\jmath^\vee : \mathcal{H} \to \mathcal{H}^{\mathcal{G}}$ extends to a surjective map $\jmath^\vee : \tilde{\mathcal{H}} \to \mathcal{H}^{\mathcal{G}}$. This map has a natural section given by*

$$\iota^\vee : \mathcal{H}^{\mathcal{G}} \to \tilde{\mathcal{H}}, \qquad \mathcal{U}_{\mathfrak{p}}^\circ \mapsto U_{\mathfrak{p}}^\circ, \ \ \mathcal{T}_{\nu,v} \mapsto T_{\iota^\vee(\nu),v}.$$

Proof. The extension is defined by $\jmath^\vee(z_v) = \mathcal{Z}_v$. It is surjective as every generator $\mathcal{T}_{\nu,v}$ and $\mathcal{U}_{\mathfrak{p}}^\circ$ is hit. One sees from the definitions that $\iota^\vee$ is a section. □

Remark 7.30. For affinoids $\Omega \subset \mathcal{W}_{\lambda_\pi}^Q$, and M an $\mathcal{H} \otimes \mathcal{O}_\Omega$-module upon which Z_v acts by $\eta_\Omega(\varpi_v)^n$, the action extends to $\tilde{\mathcal{H}} \otimes \mathcal{O}_\Omega$, where z_v acts by $\eta_\Omega(\varpi_v)$. From above, this is true for $M = \mathrm{H}_{\mathrm{c}}^t(S_K, \mathscr{D}_\Omega) \otimes_{\mathcal{O}_\Omega} \mathcal{O}_{\mathscr{I}}$, the specialisation of the cohomology to $\mathscr{I}$.

7.7.2. Eigenvariety data. By [55, Cor. 3.1.5], we may recover $\mathscr{I}$ as the eigenvariety attached to an eigenvariety datum

$$\mathcal{D} = (\mathscr{W}^Q_{0,\lambda_\pi}, \mathscr{Z}_{\mathscr{I}}, \mathscr{M}^t_{\mathscr{I}}, \mathcal{H}, \psi)$$

in the sense of Definition 3.1.1 *op. cit.* (where the Fredholm hypersurface $\mathscr{Z}_{\mathscr{I}}$ and degree t cohomology sheaf $\mathscr{M}^t_{\mathscr{I}} = \mathscr{M} \otimes \mathcal{O}_{\mathscr{I}}$ are specialised to isolate $\mathscr{I}$). We define a modified datum

$$\tilde{\mathcal{D}} = (\mathscr{W}^Q_{0,\lambda_\pi}, \mathscr{Z}_{\mathscr{I}}, \mathscr{M}^t_{\mathscr{I}}, \tilde{\mathcal{H}}, \tilde{\psi}).$$

Here $\tilde{\mathcal{H}}$ acts on $\mathscr{M}^t_{\mathscr{I}}$ by Remark 7.30 (giving $\tilde{\psi} : \tilde{\mathcal{H}} \to \mathrm{End}(\mathscr{M}^t_{\mathscr{I}})$). This gives an eigenvariety $\tilde{\mathscr{I}}$.

Lemma 7.31. *The inclusion $\mathcal{H} \hookrightarrow \tilde{\mathcal{H}}$ induces an isomorphism $\tilde{\mathscr{I}} \xrightarrow{\sim} \mathscr{I}$.*

Proof. The image of $z_v \otimes 1 \in \tilde{\mathcal{H}} \otimes \mathcal{O}_\Omega$ in $\mathrm{End}_{\mathcal{O}_\Omega}(\mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega) \otimes_{\mathcal{O}_\Omega} \mathcal{O}_{\mathscr{I}}$ is, by definition, equal to $1 \otimes \eta_\Omega(\varpi_v)$, which is also in the image of $\mathcal{H} \otimes \mathcal{O}_\Omega$. As this is the only difference between $\tilde{\mathcal{H}} \otimes \mathcal{O}_\Omega$ and $\mathcal{H} \otimes \mathcal{O}_\Omega$, they have the same image in this endomorphism ring, so the local pieces of $\mathscr{I}$ and $\tilde{\mathscr{I}}$ are the same. As the gluing data in [46, Thm. 4.2.2] depends only the local pieces, not the abstract Hecke algebra, we conclude. $\square$

Finally, as in [19, §5.2.2], at level $\mathcal{K}$ there is an eigenvariety datum

$$\mathcal{D}^{\mathcal{G}} = (\mathscr{W}^Q_{\lambda_\pi}, \mathscr{Z}^{\mathcal{G}}, \mathscr{M}^{\mathcal{G}}, \mathcal{H}^{\mathcal{G}}, \psi^{\mathcal{G}})$$

which gives the $\mathcal{Q}$-parabolic eigenvariety $\mathscr{E}^{\mathcal{G},\mathcal{Q}}_{\lambda_\pi}(\mathcal{K})$ for $\mathcal{G}$. (Note that $\jmath$, from [14, §2], identifies the $\mathcal{Q}$-parabolic weight space for $\mathcal{G}$ with the Q-parabolic weight space for G).

7.7.3. Symplectic points.

Definition 7.32. Let $y \in \mathscr{E}^Q_{\lambda_0}(K)$ be a point with corresponding eigensystem $\phi_y : \mathcal{H} \to L$. We say x is *symplectic* if there is a point $y^{\mathcal{G}} \in \mathscr{E}^{\mathcal{G},\mathcal{Q}}_{\lambda_0}(\mathcal{K})$ for some $\mathcal{K}$, such that ϕ_y factors as

$$\phi_y : \mathcal{H} \xrightarrow{\jmath^\vee} \mathcal{H}^{\mathcal{G}} \xrightarrow{\phi_y^{\mathcal{G}}} L,$$

where $\phi^{\mathcal{G}}_y$ is the eigensystem corresponding to y.

Proposition 7.33. *If $y \in \mathscr{E}^Q_{\lambda_0}(K)$ is a classical point with non-Q-critical slope and regular weight, then y is a symplectic point if and only if it is a Shalika point.*

Proof. Suppose y is a Shalika point. Let $\tilde{\pi}_y$ and $\tilde{\Pi}_y$ be as described after the statement of Theorem 7.26; then $\phi_y = \phi_{\tilde{\pi}_y}$. By compatibility of Langlands functoriality (at $v \nmid p$) and [14, Prop. 3.7] (at $\mathfrak{p}|p$) $\phi_{\tilde{\pi}_y}$ factors as

$$\phi_{\tilde{\pi}_y} : \mathcal{H} \xrightarrow{\jmath^\vee} \mathcal{H}^{\mathcal{G}} \xrightarrow{\phi_{\tilde{\Pi}_y}} L. \tag{7.13}$$

It remains to show $\tilde{\Pi}_y$ appears in an eigenvariety for $\mathcal{G}$. Let $\mathcal{K} \subset \mathcal{G}(\mathbf{A}_{F,f})$ be open compact as above (maximal hyperspecial at $v \notin S \cup \{\mathfrak{p}|p\}$, parahoric at $\mathfrak{p}|p$) such that $\Pi^{\mathcal{K}}_y \neq \{0\}$. By [14, §3.5], the refinement $\tilde{\Pi}_y$ has non-$\mathcal{Q}$-critical slope, so by [19, Prop. 5.8], yields a point $y^{\mathcal{G}} \in \mathscr{E}^{\mathcal{G},\mathcal{Q}}_{\lambda_\pi}(\mathcal{K})$ corresponding to $\phi_{\tilde{\Pi}_y}$, and y is symplectic.

Conversely, suppose y is symplectic; then by [14, §3.5], $y^{\mathcal{G}}$ is non-$\mathcal{Q}$-critical slope in $\mathscr{E}^{\mathcal{G},\mathcal{Q}}_{\lambda_0}(\mathcal{K})$. Using regular weight, as in the proof of [19, Prop. 5.15], $y^{\mathcal{G}}$ is classical cuspidal, corresponding to some RACAR Π_y of $\mathcal{G}(\mathbf{A}_F)$. At $v \notin S \cup \{\mathfrak{p}|p\}$, $\Pi_{y,v}$ is unramified; by considering the Satake parameters and using [4, §6] we see that $\pi_{y,v}$ is the functorial transfer of $\Pi_{y,v}$. By [5] this ensures π_y is globally the transfer of Π_y. Thus π_y admits a Shalika model, as required. $\square$

Theorem 7.34. *Let $\mathscr{I} \subset \mathscr{E}^{Q}_{\lambda_0}(K)$ be an irreducible component satisfying the conditions of Theorem 7.26. Then every point of $\mathscr{I}$ is a symplectic point.*

Proof. We maintain the notation from the proof of Proposition 7.33. By [5, Prop. 5.1], which controls the image of functorial transfer at ramified places, we may choose $\mathcal{K} \subset \mathcal{G}(\mathbf{A}_{F,f})$ as above such that $\Pi_y^{\mathcal{K}} \neq \{0\}$ for all such y; we work at this level for $\mathcal{G}$.

We have the following 'inverse' of (7.13); extend $\phi_{\tilde{\pi}_y}$ to $\tilde{\phi}_{\tilde{\pi}_y} : \tilde{\mathcal{H}} \to L$ by sending $z_v \mapsto \eta_0(\varpi_v)|\varpi_v|^{\mathsf{w}_y}$. As $\iota^\vee$ is a section of $\jmath^\vee$, $\phi_{\tilde{\Pi}_y}$ factors as

$$\phi_{\tilde{\Pi}_y} : \mathcal{H}^{\mathcal{G}} \xrightarrow{\iota^\vee} \tilde{\mathcal{H}} \xrightarrow{\tilde{\phi}_{\tilde{\pi}_y}} L. \tag{7.14}$$

As in the proof of Proposition 7.16, a neighbourhood $\mathscr{U}$ of the given point $\tilde{\pi}$ contains a Zariski-dense set of non-Q-critical slope (η_0, ψ)-Shalika points $y \in \mathscr{U}$. By [37, Lem. 2.2.3], this set is also Zariski-dense in $\mathscr{I}$. By Proposition 7.33, we have associated points $y^{\mathcal{G}} \in \mathscr{E}^{\mathcal{G},\mathcal{Q}}_{\lambda_0}(\mathcal{K})$.

Let $\tilde{\mathscr{I}}^\circ$ denote the nilreduction of $\tilde{\mathscr{I}}$. By [55, Thm. 3.2.1] and (7.14), the map $\iota^\vee$ induces a map $g : \tilde{\mathscr{I}}^\circ \to \mathscr{E}^{\mathcal{G},\mathcal{Q}}_{\lambda_\pi}(\mathcal{K})$ interpolating the association $y \mapsto y^{\mathcal{G}}$ for the Zariski-dense set of (η_0, ψ)-Shalika points $y \in \mathscr{I}$. Conversely let $\mathscr{I}^{\mathcal{G}}$ be the irreducible component containing $g(\tilde{\mathscr{I}}^\circ)$, and $\mathscr{I}^{\mathcal{G},\circ}$ its nilreduction. By the same theorem and (7.13), $\jmath^\vee$ induces a map $f : \mathscr{I}^{\mathcal{G},\circ} \to \mathscr{I}$.

As nilreductions do not change closed points, and $\tilde{\mathscr{I}}$ is isomorphic to $\mathscr{I}$ by Lemma 7.31, the maps f and g induce inverse bijections on the sets of closed points in $\mathscr{I}$ and $\mathscr{I}^{\mathcal{G}}$. By [55, Thm. 3.2.1] again this means every eigensystem in $\mathscr{I}$ factors through $\jmath^\vee$, and hence is symplectic. □

Theorem 7.26 follows immediately by combining Theorem 7.34 with Proposition 7.33.

8. p-adic L-functions over the eigenvariety

Finally we construct p-adic L-functions in families and prove Theorem C of the introduction. To do so, we pursue a similar overall strategy to that in single weights. In the single weight situation, namely row (M) of Figure 1 of the introduction, we:

(1) replaced $\mathscr{V}_\lambda^\vee$ with a larger coefficient sheaf $\mathscr{D}_\lambda$ and found a single evaluation map $\mathrm{Ev}_\lambda : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\lambda) \to \mathcal{D}(\mathrm{Gal}_p, \overline{\mathbf{Q}}_p)$ interpolating all the $\mathrm{Ev}_{\chi,j}$;

(2) exhibited a canonical eigenclass $\Phi_{\tilde{\pi}} \in \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\lambda)$, constructed via a choice of Friedberg–Jacquet test vector $W_f^{\mathrm{FJ}} \in \mathcal{S}^{\eta_f}_{\psi_f}(\pi_f)$.

In the top row (T) of Figure 1, we instead work over a Shalika family $\mathscr{C}$ in the eigenvariety, and we need to:

(1′) replace $\mathscr{D}_\lambda$ with $\mathscr{D}_\Omega$ and construct an evaluation map $\mathrm{Ev}_\Omega : \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega) \to \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_\Omega)$ interpolating the maps Ev_λ as λ varies in Ω;

(2′) exhibit an eigenclass $\Phi_{\mathscr{C}} \in \mathrm{H}^t_{\mathrm{c}}(S_K, \mathscr{D}_\Omega)$ over $\mathscr{C}$ interpolating, up to $\overline{\mathbf{Q}}_p^\times$, the eigenclasses $\Phi_{\tilde{\pi}_y}$ as y varies over Shalika points of $\mathscr{C}$.

This would give the top commutative square in Figure 1. We have already handled part (1′) in §6, and in this section, we explain how to obtain the eigenclass $\Phi_{\mathscr{C}}$ of (2′).

In §7, we proved existence and étaleness of Shalika families, but had to consider and compare two separate levels $K(\tilde{\pi})$ and $K_1(\tilde{\pi})$ to do so. To vary p-adic L-functions over these families requires more precise control still, since for (2′) we must show not only that $\tilde{\pi}$ varies in a Shalika family, but that specific vectors inside these representations – the cusp forms W_f^{FJ} from §2.10 – also vary p-adic analytically in this family. In this chapter, we prove such variation if

π satisfies an automorphic hypothesis (Hypothesis 8.6). This is captured in Theorem 8.11, a modification/strengthening of Theorem 7.6 tailored for variation of p-adic L-functions. This is Theorem B′ of the introduction, and is proved in §8.3.

Hypothesis 8.6 is automatic in tame level 1 (that is, when π is spherical at all $v \nmid p\infty$), so our results are unconditional in this case.

In §8.4, we use Theorem 8.11 to construct the multi-variable p-adic L-function and prove Theorem C. Finally in §8.5 we give an application of this construction; suppose $\tilde{\pi}$ satisfies our running assumptions, and is non-Q-critical but has Q-critical slope. In this case the slope condition of Proposition 6.25 does not apply, and we could not previously show that $\mathcal{L}_p(\tilde{\pi})$ was uniquely determined. In §8.5, we show that when the multi-variable p-adic L-function exists, $\mathcal{L}_p(\tilde{\pi})$ is uniquely determined by interpolation over the family.

An unconditional treatment of higher tame level would require new input from local representation theory. We describe these representation-theoretic obstructions in §8.1, and give our hypothesis to relax the tame level 1 restriction in §8.2. Since these problems are of a very different nature to the methods developed in the rest of this paper, we do not attempt to prove this hypothesis here.

8.1. On the choice of local test vectors. Suppose $\tilde{\pi}$ satisfies (C1-2) of Conditions 2.8. Recall from §2.4 that

$$S = \{v \nmid p\infty : \pi_v \text{ ramified}\}.$$

To vary the cusp form $W_f^{\mathrm{FJ}} = \otimes_v W_v^{\mathrm{FJ}}$ in a p-adic family, we need control on the local vectors W_v^{FJ}. For this, there are three natural cases:

- $v = \mathfrak{p} \mid p$. At such v, the choice of local vector $W_v^{\mathrm{FJ}} = W_{\mathfrak{p}}$ is prescribed by the choice of Q-refinement in condition (C2) of Conditions 2.8.
- $v \nmid p\infty$, and $v \notin S$ (i.e. π_v is spherical). In this case, we have complete control over the choice of W_v^{FJ}: as outlined in §2.6, the spherical vector in $\mathcal{S}_{\psi_v}^{\eta_v}(\pi_v)$ is a Friedberg–Jacquet test vector, i.e.

$$\zeta_v\left(s+\tfrac{1}{2}, W_v^{\mathrm{FJ}}, \chi_v\right) = [\mathrm{N}_{F/\mathbf{Q}}(v)^s\chi_v(\varpi_v)]^{n\delta_v} \cdot L\left(\pi_v \otimes \chi_v, s+\tfrac{1}{2}\right). \tag{8.1}$$

- $v \nmid p\infty$ and $v \in S$ (i.e. π_v is ramified). In this case, the choice of Friedberg–Jacquet test vector W_v^{FJ} is not well-understood; the proof of its existence is not constructive.

When $S = \varnothing$ (that is, we are in the case of tame level 1), this means we have control on W_v^{FJ} at every finite place v. In this case, we have

$$K(\tilde{\pi}) = K_1(\tilde{\pi}) = \prod_{\mathfrak{p}|p} J_{\mathfrak{p}} \prod_{v\nmid p\infty} \mathrm{GL}_{2n}(\mathcal{O}_v).$$

Crucially, this means:

Proposition 8.1. *Let $\tilde{\pi}$ be a non-Q-critical Q-refined RACAR satisfying (C1-2) of Conditions 2.8. Suppose π has tame level 1. Then*

(1) $\mathcal{S}_{\psi_f}^{\eta_f}\left(\pi_f^{K(\tilde{\pi})}\right)[\![U_{\mathfrak{p}} - \alpha_{\mathfrak{p}} : \mathfrak{p}|p]\!]$ is a line, and

(2) this line has a generator

$$W_f^{\mathrm{FJ}} = \otimes_{\mathfrak{p}|p} W_{\mathfrak{p}} \otimes_{v\nmid p\infty} W_v^{\mathrm{FJ}},$$

where each $W_{\mathfrak{p}}$ is as in (C2) and each W_v^{FJ} is a Friedberg–Jacquet test vector.

Later in this chapter, we show that these two properties imply W_f^{FJ} varies analytically over the eigenvariety from §7. This allows us to construct a p-adic L-function over any tame level 1 eigenvariety.

For ramified π_v, finding an explicit Friedberg–Jacquet test vector W_v^{FJ} is an interesting and difficult research question in local representation theory, of a very different flavour to the p-adic methods and results of the present paper. To generalise our constructions to higher tame level, at $v \in S$ we want to show something like:

- (1′) there is an explicit open compact subgroup $K_v \subset \mathrm{GL}_{2n}(\mathcal{O}_v)$, and a 1-dimensional subspace of $\pi_v^{K_v}$ cut out as the generalised eigenspace of a family of Hecke operators;
- (2′) the image of a generator of this line under $\mathcal{S}_{\psi_v}^{\eta_v}$ is a Friedberg–Jacquet test vector in the Shalika model for π_v.

The theory of Whittaker new vectors, as described in (7.1), gives an unconditional source of explicit K_v such that (1′) is satisfied (without needing to use any Hecke operators). Evidence of some natural compatibility between the Whittaker and Shalika models is provided by a recent result of Grobner–Matringe [44]. There, it is shown that if η_v is unramified, and $W_v \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi_v)$ is fixed by $K_{1,v}(m(\pi_v))$ – that is, W_v is the image of a Whittaker new vector in the Shalika model – then $W_v(1_{2n}) \neq 0$. The natural question of whether such a W_v is a Friedberg–Jacquet test vector, however, appears very difficult to check.

8.2. Shalika new vectors. It is natural to ask if there is a Shalika analogue of the theory of Whittaker new vectors. In §8.2.1, we introduce a theory of Shalika new vectors, and in §8.2.2, give examples to show our theory is non-empty. In §8.2.3 we hypothesise that Shalika new vectors are Friedberg–Jacquet test vectors.

8.2.1. Shalika conductors. Let $c \geqslant 1$ be an integer. Rather than the subgroups $K_{1,v}(c)$ used in the Whittaker theory, we consider the 'Q-parahoric' analogue

$$\begin{aligned} J_v(c) &:= \begin{pmatrix} \mathrm{GL}_n(\mathcal{O}_v) & M_n(\mathcal{O}_v) \\ \varpi_v^c \cdot M_n(\mathcal{O}_v) & \mathrm{GL}_n(\mathcal{O}_v) \end{pmatrix} \\ &= \left\{ g \in \mathrm{GL}_{2n}(\mathcal{O}_v) \;\middle|\; g \,(\mathrm{mod}\, \varpi_v^c) \in Q(\mathcal{O}_v/\varpi_v^c) \right\}. \end{aligned}$$

We also set $J_v(0) = \mathrm{GL}_{2n}(\mathcal{O}_v)$. Note that $J_v(1)$ is just the parahoric subgroup J_v.

Definition 8.2. Suppose π_v is an irreducible admissible representation of $\mathrm{GL}_{2n}(F_v)$ that admits an (η_v, ψ_v)-Shalika model.

(1) The *Shalika conductor* $c(\pi_v)$ of π_v is the smallest $c \in \mathbf{Z}_{\geqslant 0}$ (if it exists) such that

$$\mathcal{S}_{\psi_v}^{\eta_v}\big(\pi_v^{J_v(c),\eta_v}\big) := \left\{ W_v \in \mathcal{S}_{\psi_v}^{\eta_v}(\pi_v) \;\middle|\; W_v(-\cdot k) = \eta_v(\det(k_2)) W_v(-) \;\; \forall k = \left(\begin{smallmatrix} k_1 & * \\ * & k_2 \end{smallmatrix}\right) \in J_v(c) \right\} \neq \{0\}.$$

(2) If $\mathcal{S}_{\psi_v}^{\eta_v}\big(\pi_v^{J_v(c(\pi_v)),\eta_v}\big)$ is a line, we call a *Shalika new vector* any generator of this line.

Lemma 8.3. *Suppose π_v is an irreducible admissible representation of $\mathrm{GL}_{2n}(F_v)$ that admits an (η_v, ψ_v)-Shalika model. Then the Shalika conductor $c(\pi_v) \in \mathbf{Z}_{\geqslant 0}$ exists.*

Moreover for any $c \geqslant c(\pi_v)$ one has $\dim(\pi_v^{J_v(c+1),\eta_v}) > \dim(\pi_v^{J_v(c),\eta_v})$.

Proof. As π_v admits an (η_v, ψ_v)-Shalika model, the Friedberg–Jacquet linear functional [41] is a non-zero element of $\mathrm{Hom}_{H(\mathcal{O}_v)}(\pi_v, \eta_v)$. As $H(\mathcal{O}_v)$ is compact, one therefore has $\mathrm{Hom}_{H(\mathcal{O}_v)}(\eta_v, \pi_v) \neq \{0\}$; that is, there exists a non-zero vector $\varphi \in \pi_v^{H(\mathcal{O}_v),\eta_v}$.

As φ is smooth, there exists some $c \gg 0$ such that φ is fixed by $N_Q(\varpi_v^c\mathcal{O}_v)$ and $N_Q^-(\varpi_v^c\mathcal{O}_v)$. Let $t_v = \mathrm{diag}(\varpi_v I_n, I_n)$. Since $J_v(2c) = t_v^{-c}N_Q(\varpi_v^c\mathcal{O}_v)H(\mathcal{O}_v)N_Q^-(\varpi_v^c\mathcal{O}_v)t_v^c$, we deduce $t_v^{-c}\cdot\varphi \in \pi_v^{J_v(2c),\eta_v}$. Thus for $c \gg 0$, the space in Definition 8.2(1) is non-zero, so the conductor exists.

For the proof of the last claim, to ease notation, we will drop η_v from the exponent. As $\pi_v^{J_v(c)} \subset \pi_v^{J_v(c+1)}$, it suffices to prove that the inclusion is strict.

- If $c = 0$, then π_v is spherical, and $\dim(\pi_v^{J_v(1)}) = \binom{2n}{n} > 1$ (see e.g. [38, §3.1]).

If $c \geqslant 1$, let φ be an element of $\pi_v^{J_v(c)}$ which we can inductively assume not in $\pi_v^{J_v(c-1)}$. Suppose that $\pi_v^{J_v(c+1)} = \pi_v^{J_v(c)}$. Note $t_v^{-1}\cdot\varphi \in {\pi_v}^{t_v^{-1}J_v(c)t_v}$, and

$$J_v(c+1) \subset t_v^{-1}\cdot J_v(c)\cdot t_v,$$

hence

$$t_v^{-1}\cdot\varphi \in \pi_v^{J_v(c+1)} = \pi_v^{J_v(c)}.$$

We thus deduce

$$\varphi \in \pi_v^{t_vJ_v(c)t_v^{-1}},$$

so φ is fixed by both $J_v(c)$ and $t_vJ_v(c)t_v^{-1}$, and hence by the group $J' \subset \mathrm{GL}_{2n}(F_v)$ that they generate. To obtain a contradiction with the assumption that $\varphi \notin {\pi_v}^{J_v(c-1)}$, it suffices to show

$$J_v(c-1) \subset J'. \tag{8.2}$$

- Suppose $c \geqslant 2$. Then $J_v(c-1)$ admits a parahoric decomposition $J_v(c-1) = [J_v(c-1)\cap N_Q^-(\mathcal{O}_v)]\cdot H(\mathcal{O}_v)N_Q(\mathcal{O}_v) = t_v[J_v(c)\cap N_Q^-(\mathcal{O}_v)]t_v^{-1}\cdot H(\mathcal{O}_v)N_Q(\mathcal{O}_v)$. This lies in $t_vJ_v(c)t_v^{-1}\cdot J_v(c) \subset J'$, as required.
- If $c = 1$, then observe that

  $$H(\mathcal{O}_v)N_Q(\mathcal{O}_v) \subset J_v(1) \subset J'$$

  and

  $$N_Q^-(\mathcal{O}_v) = t_vN_Q^-(\varpi_v\mathcal{O}_v)t_v^{-1} \subset t_vJ_v(1)t_v^{-1} \subset J',$$

  hence

  $$\begin{pmatrix} I_{n-1} & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & I_{n-1} \end{pmatrix} = \begin{pmatrix} I_{n-1} & & \\ & \begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & I_{n-1} \end{pmatrix}\begin{pmatrix} I_{n-1} & & \\ & \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} & \\ & & I_{n-1} \end{pmatrix}$$

  Both elements in the product are in (**??**), so this element lies in J'', and we are done. ☐

We put the theory of Shalika new vectors in a form closer to Proposition 8.1(1). For $c = c(\pi_v)$, let

$$J_v^\eta(c) := \ker[\eta_v \circ \det_2 : J_v(c) \to \mathbf{C}^\times]$$

and consider the (diamond) Hecke operators

$$S_{\alpha_v} = [J_v^\eta(c)\ \mathrm{diag}(1,\dots,1,\alpha_v)\ J_v^\eta(c)], \qquad \alpha_v \in \mathcal{O}_v^\times.$$

Lemma 8.4. *Any π_v as in Lemma 8.3 admits a Shalika new vector if and only if*

$$\dim_{\mathbf{C}}\ {\pi_v}^{J_v^\eta(c(\pi_v))}\big[S_{\alpha_v} - \eta_v(\alpha_v) : \alpha_v \in \mathcal{O}_v^\times\big] = 1. \tag{8.3}$$

Proof. For $\alpha_v \in \mathcal{O}_v^\times$, let $t_{\alpha_v} = \mathrm{diag}(1,...,1,\alpha_v)$. Via $\det_2$, one sees that $\{t_{\alpha_v} : \alpha_v \in \mathcal{O}_v^\times\}$ contains a complete set of representatives for $J_v(c(\pi_v))/J_v^\eta(c(\pi_v))$. Additionally the Hecke operator S_{α_v} is simply right translation by t_{α_v}. Hence (8.3) is a reformulation of Definition 8.2(2). ☐

8.2.2. Shalika new vectors for parahoric-spherical representations. If π_v is spherical, then it has Shalika conductor 0, and a spherical vector is a Shalika new vector. The following lemma shows that both possibilities arise even in the simplest case when $n = 4$ and π_v is parahoric-spherical, i.e. has non-zero vectors fixed by the parahoric subgroup J_v. We are indebted to David Loeffler for having drawn to our attention that such counter-examples exist, and to Andrei Jorza for having helped us find some (positive) examples.

Lemma 8.5. *Let* St_v *denote the Steinberg representation of* $\mathrm{GL}_2(F_v)$.

- *(i) Let* π_v *be the full parabolic induction from* $Q(F_v)$ *to* $\mathrm{GL}_4(F_v)$ *of* $\mathrm{St}_v \times \mathrm{St}_v$. *Then* π_v *is parahoric-spherical and admits a Shalika new vector.*
- *(ii) Let* P *denote the* $(1, 2, 1)$ *parabolic of* GL_4 *and let* π'_v *be the full parabolic induction from* $P(F_v)$ *to* $\mathrm{GL}_4(F_v)$ *of* $\mathbf{1} \times \mathrm{St}_v \times \mathbf{1}$. *Then* π'_v *is parahoric spherical but does not admit a Shalika new vector.*

Proof. Let us first observe that both π_v and π'_v are ramified representations admitting a Shalika model for $\eta_v = \mathbf{1}$. We realise the Weyl groups W_Q and W_P as the subgroups of the Weyl group S_4 of GL_4 generated respectively by $\{(12), (34)\}$ and $\{(23)\}$. Let $\{\beta_1, \beta_2, \beta_3\}$ denote the simple roots of GL_4. Note the parabolic subgroups Q and P correspond to the subsets $\{\beta_1, \beta_3\}$ and $\{\beta_2\}$ respectively.

(i) It suffices to show that $\dim_{\mathbf{C}} \pi_v^{J_v} = 1$. One can easily check that a set of representatives of the double coset $W_Q\backslash S_4/W_Q$ is given by $\overline{W} = \{(1), (23), (13)(24)\}$. By [39, §1] the dimension of $\pi_v^{J_v}$ is given by the number of $w \in \overline{W}$ such that $w \cdot \{\beta_1, \beta_3\} \cap \{\beta_1, \beta_3\} = \varnothing$. This is only the case for $w = (23)$.

(ii) It suffices to show that $\dim_{\mathbf{C}} \pi_v^{J_v} > 1$. A set of representatives of the double coset $W_P\backslash S_4/W_Q$ is given by $\overline{W}' = \{(1), (123), (1243), (243)\}$. In this case, there are two elements $w \in \overline{W}'$, namely (1) and (1243), for which $w \cdot \{\beta_1, \beta_3\} \cap \{\beta_2\} = \varnothing$. The same argument as above shows that the space of J_v-invariants in π'_v is 2-dimensional. □

We refer the interested reader to [39, §1] for a full classification of the parahoric-spherical generic representations of GL_{2n} admitting a Shalika model.

8.2.3. A hypothesis on Shalika new vectors. Given the above theory of Shalika new vectors, it seems natural to make the following hypothesis.

Hypothesis 8.6. *Let* $c \in \mathbf{Z}_{\geqslant 0}$. *For any* π_v *admitting a Shalika new vector of conductor* c*, a multiple of the latter is also a Friedberg–Jacquet test vector for* π_v *(as in §2.6).*

As evidence towards this, we note that Friedberg–Jacquet [41, Prop. 3.2] proved that Hypothesis 8.6 holds for $c = 0$ (see also §2.6). Further, in [39, §1] it is shown that the π_v admitting a Shalika new vector of conductor $c = 1$ are precisely the parahoric-spherical representations which are maximally Steinberg, and further, it is established in [39, §2] that for such π_v Hypothesis 8.6 holds provided that π_v is regular (i.e., occurs in $\mathrm{Ind}_B^G \theta_v$ with θ_v regular).

8.3. Shalika families, refined.

8.3.1. Set-up: Shalika Hecke algebra and the eigenvariety $\mathscr{E}^S$. Let $\tilde{\pi}$ be a non-Q-critical Q-refined RACAR of weight λ_π satisfying (C1-2) of Conditions 2.8. Recall $S = \{v \nmid p\infty : \pi_v \text{ ramified}\}$. For the rest of this chapter, we assume that:

for all $v \in S$, the local representation π_v admits a Shalika new vector of conductor $c(\pi_v)$, and Hypothesis 8.6 holds for $c = c(\pi_v)$.

We emphasise again that this assumption is empty when π has tame level 1.

Given these assumptions, for the rest of the paper we fix a sufficiently large coefficient field $L/\mathbf{Q}_p$ as in §2.10, and drop it from notation. We also fix a precise choice of level subgroup

$$K(\tilde{\pi}) = \prod_{\mathfrak{p}|p} J_{\mathfrak{p}} \cdot \prod_{v\in S} J_v^{\eta}(c(\pi_v)) \prod_{v\notin S\cup\{\mathfrak{p}|p\}} \mathrm{GL}_{2n}(\mathcal{O}_v) \subset G(\mathbf{A}_f). \tag{8.4}$$

Recall the Hecke alegbra $\mathcal{H}$ from Definition 2.9. In light of Lemma 8.4, it is necessary to modify our Hecke algebra by adding the diamond operators S_{α_v}.

Definition 8.7. The *Shalika Hecke algebra of level* $K(\tilde{\pi})$ is

$$\mathcal{H}^{\mathcal{S}} := \mathcal{H}[S_{\alpha_v} : v \in S, \alpha_v \in \mathcal{O}_v^\times].$$

This acts on $\pi^{K(\tilde{\pi})}$ and $\mathrm{H}_c^\bullet(S_{K(\tilde{\pi})}, -)$.

We define some modified objects exactly analogous to their incarnations in §7, but with $\mathcal{H}^{\mathcal{S}}$ replacing $\mathcal{H}$. Recall $\psi_{\tilde{\pi}}$ from Definition 2.9 (noting $E \subset L$).

Definition 8.8. Define a character $\psi_{\tilde{\pi}}^{\mathcal{S}} : \mathcal{H}^{\mathcal{S}} \otimes L \longrightarrow L$ extending $\psi_{\tilde{\pi}}$ by sending $S_{\alpha_v} \mapsto \eta(\alpha_v)$. Let $\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}} := \ker(\psi_{\tilde{\pi}}^{\mathcal{S}})$ be the associated maximal ideal. For $\Omega \subset \mathscr{W}_{\lambda_\pi}^Q$ a neighbourhood of λ_π, as in (7.4) we get an associated maximal ideal, also denoted $\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}$, in $\mathcal{H}^{\mathcal{S}} \otimes \mathcal{O}_\Omega$.

Proposition 8.9. *Let $\tilde{\pi}$ satisfy Conditions 2.8 and assume Hypothesis 8.6 for $v \in S$. For any $\epsilon \in \{\pm 1\}^\Sigma$,*

$$\dim_L \mathrm{H}_c^t(S_{K(\tilde{\pi})}, \mathscr{V}_{\lambda_\pi}^\vee)^\epsilon_{\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}} = 1.$$

If $\tilde{\pi}$ is non-Q-critical, then

$$\dim_L \mathrm{H}_c^t(S_{K(\tilde{\pi})}, \mathscr{D}_{\lambda_\pi})^\epsilon_{\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}} = 1.$$

Proof. As in Proposition 7.18, the $\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}$-torsion in $\pi_f^{K(\tilde{\pi})}$ is a line; for $v \in S$, this is by Lemma 8.4 and the assumed existence of a Shalika new vector. We conclude

$$\dim_L \mathrm{H}_c^t(S_{K(\tilde{\pi})}, \mathscr{V}_{\lambda_\pi}^\vee)^\epsilon_{\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}} = 1$$

exactly as in Proposition 7.18. If $\tilde{\pi}$ is non-Q-critical, then the rest follows as

$$\mathrm{H}_c^\bullet(S_{K(\tilde{\pi})}, \mathscr{D}_{\lambda_\pi})_{\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}} \cong \mathrm{H}_c^\bullet(S_{K(\tilde{\pi})}, \mathscr{V}_{\lambda_\pi}^\vee)_{\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}}$$

as K_∞/K_∞°-modules. □

Via §3.3, let Ω be an affinoid neigbourhood of λ_π in $\mathscr{W}_{\lambda_\pi}^Q$ such that $\mathrm{H}_c^t(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)$ admits a slope $\leqslant h$ decomposition with respect to the U_p operator.

Definition 8.10.

- Let $\mathbf{T}_{\Omega,h}^{\mathcal{S}}$ be the image of the natural map

$$\mathcal{H}^{\mathcal{S}} \otimes \mathcal{O}_\Omega \longrightarrow \mathrm{End}_{\mathcal{O}_\Omega}\big(\mathrm{H}_c^t(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^{\leqslant h}\big).$$

- Let $\mathscr{E}_{\Omega,h}^{\mathcal{S}} := \mathrm{Sp}(\mathbf{T}_{\Omega,h}^{\mathcal{S}})$.
- For $\epsilon \in \{\pm 1\}^\Sigma$, write $\mathbf{T}_{\Omega,h}^{\mathcal{S},\epsilon}$ and $\mathscr{E}_{\Omega,h}^{\mathcal{S},\epsilon}$ when using only ϵ-parts of the cohomology.
- For a classical cuspidal point $y \in \mathcal{E}_{\Omega,h}^{\mathcal{S}}$ (for terminology as in §7.1), we write $\mathfrak{m}_y^{\mathcal{S}} := \mathfrak{m}_{\pi_y}^{\mathcal{S}}$.

When π has tame level 1, then $\mathcal{H} = \mathcal{H}^{\mathcal{S}}$, $\mathfrak{m}_{\tilde{\pi}} = \mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}$, $\mathscr{E}_{\Omega,h}^{\mathcal{S},\epsilon} = \mathscr{E}_{\Omega,h}^\epsilon$, etc., so these are all the exactly the same objects as in §7.

8.3.2. Shalika families, refined: statement. The following is a more precise version of Theorem B$'$, refining Theorem 7.6. We recall that we have fixed the level subgroup $K(\tilde{\pi})$ and coefficient field L (in §8.3.1), and we drop both from further notation. Let $\alpha_p^\circ = \prod_{\mathfrak{p}|p}(\alpha_{\mathfrak{p}}^\circ)^{e_{\mathfrak{p}}}$, and fix $h \geqslant v_p(\alpha_p^\circ)$ and $\epsilon \in \{\pm 1\}^\Sigma$. By Proposition 8.9, $\tilde{\pi}$ contributes to $\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_{\lambda_\pi})^{\leqslant h}$.

Theorem 8.11. *Let $\tilde{\pi}$ be non-Q-critical satisfying (C1-2), and suppose that π admits a non-zero Deligne-critical L-value at p (Definition 7.3), that λ_π is H-regular* (7.3), *and that $\tilde{\pi}$ is strongly non-Q-critical (Definition 3.14). Suppose that for all $v \in S$:*

- *π_v admits a Shalika new vector of conductor $c(\pi_v)$, and*
- *Hypothesis 8.6 holds for $c = c(\pi_v)$.*

Then there exists a point $x_{\tilde{\pi}}^{\mathcal{S}}$ attached to $\tilde{\pi}$ in $\mathscr{E}_{\Omega,h}^{\mathcal{S}}$. Let $\mathscr{C}$ be the connected component of $\mathscr{E}_{\Omega,h}^{\mathcal{S}}$ through $x_{\tilde{\pi}}^{\mathcal{S}}$. Then, after possibly shrinking Ω,

- **(étaleness)** *the weight map $\mathscr{C} \to \Omega$ is étale,*
- **(density of Shalika points)** *$\mathscr{C}$ contains a Zariski-dense set $\mathscr{C}_{\mathrm{nc}}$ of classical cuspidal points y corresponding to non-Q-critical Q-refined RACARs $\tilde{\pi}_y$ satisfying (C1-2) of Conditions 2.8,*
- **(Shalika new vectors in families)** *for each $y \in \mathscr{C}_{\mathrm{nc}}$ and for all $v \in S$, $\pi_{y,v}$ admits a Shalika new vector of conductor $c(\pi_{y,v}) = c(\pi_v)$, and*
- **(family of eigenclasses)** *for each $\epsilon \in \{\pm 1\}^\Sigma$, there exists a Hecke eigenclass $\Phi_{\mathscr{C}}^\epsilon \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^\epsilon$ such that for every $y \in \mathscr{C}_{\mathrm{nc}}$ with $w(y) = \lambda_y$, the specialisation $\mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^\epsilon)$ generates $\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_{\lambda_y})^\epsilon_{\mathfrak{m}_y^{\mathcal{S}}}$.*

The non-vanishing, H-regular and strongly non-Q-critical hypotheses hold if $\tilde{\pi}$ has non-Q-critical slope and λ_π is regular (Theorem 3.16, Lemma 7.4), so this implies Theorem B$'$ of the introduction. The proof of Theorem 8.11 will occupy the rest of §8.3; it is similar to the methods of §7, with the addition of some standard arguments, which we highlight.

8.3.3. Cyclicity results. We now prove an analogue of Proposition 7.19 and delocalise to a neighbourhood. Let $\mathbf{T}_{\Omega,\tilde{\pi}}^{\mathcal{S},\epsilon} = (\mathbf{T}_{\Omega,h}^{\mathcal{S},\epsilon})_{\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}}$ denote the localisation of $\mathbf{T}_{\Omega,h}^{\mathcal{S},\epsilon}$ at $\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}$. Recall $\Lambda = \mathcal{O}_{\Omega,\mathfrak{m}_{\lambda_\pi}}$.

The following is the only place in the proof where we use the existence of Shalika new vectors at $v \in S$ (via Proposition 7.18).

Proposition 8.12. *(i) There exists a proper ideal $I_{\tilde{\pi}} \subset \Lambda$ such that*

$$\mathbf{T}_{\Omega,\tilde{\pi}}^{\mathcal{S},\epsilon} \cong \Lambda/I_{\tilde{\pi}}.$$

(ii) The space $\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^\epsilon_{\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}}$ is free of rank one over $\mathbf{T}_{\Omega,\tilde{\pi}}^{\mathcal{S},\epsilon}$.

Proof. Using Proposition 8.9, and arguing exactly as in Proposition 7.19, there exists $I_{\tilde{\pi}}$ such that

$$\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^\epsilon_{\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}} \cong \Lambda/I_{\tilde{\pi}},$$

and hence (i) follows. The actions of Λ and $\mathbf{T}_{\Omega,\tilde{\pi}}^{\mathcal{S},\epsilon}$ are compatible, so $\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_\Omega)^\epsilon_{\mathfrak{m}_{\tilde{\pi}}^{\mathcal{S}}}$ is free of rank one over $\mathbf{T}_{\Omega,\tilde{\pi}}^{\mathcal{S},\epsilon}$, giving (ii). ☐

Since $\mathrm{H}^t_c(S_{K(\tilde\pi)},\mathscr{D}_\Omega)_{\mathfrak{m}^{\mathcal{S}}_{\tilde\pi}} \neq 0$, there exists a point $x^{\mathcal{S},\epsilon}_{\tilde\pi} \in \mathcal{E}^{\mathcal{S},\epsilon}_{\Omega,h}$ attached to $\tilde\pi$.

To construct the eigenclasses $\Phi^\epsilon_{\mathscr{C}}$ of Theorem 8.11, we want to delocalise Proposition 8.12 to a neighbourhood of $x^{\mathcal{S},\epsilon}_{\tilde\pi}$ in $\mathscr{E}^{\mathcal{S},\epsilon}_{\Omega,h}$. For this, it is convenient to work with rigid analytic localisations instead of the algebraic localisations we have considered thus far. Define

$$\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}} = \varinjlim_{x^{\mathcal{S},\epsilon}_{\tilde\pi} \in C \subset \mathscr{E}^{\mathcal{S},\epsilon}_{\Omega,h}} \mathcal{O}_C, \qquad \text{and} \qquad \Lambda_{\lambda_\pi} = \varinjlim_{\lambda_\pi\in\Omega'\subset\Omega} \mathcal{O}_{\Omega'},$$

where the limits are over open affinoid neighbourhoods. These are the local rings of the rigid spaces $\mathscr{E}^{\mathcal{S},\epsilon}_{\Omega,h}$ and Ω at $x^{\mathcal{S},\epsilon}_{\tilde\pi}$ and λ_π respectively. By [27, §7.3.2], the rigid localisation $\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}}$ (resp. Λ_{λ_π}) is a faithfully flat algebra over the algebraic localisation $\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\tilde\pi}$ (resp. Λ), and the natural map induces an isomorphism $\widehat{\mathbf{T}}^{\mathcal{S},\epsilon}_{\Omega,\tilde\pi} \cong \widehat{\mathbf{T}}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}}$ (resp. $\widehat\Lambda \cong \widehat\Lambda_{\lambda_\pi}$) of their completions.

Proposition 8.13. *(i) There is an ideal $I_{x_{\tilde\pi}} \subset \Lambda_{\lambda_\pi}$ such that*

$$\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}} \cong \Lambda_{\lambda_\pi}/I_{x_{\tilde\pi}}.$$

(ii) Let

$$\mathscr{C}^\epsilon = \mathrm{Sp}(\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\mathscr{C}}) \subset \mathscr{E}^{\mathcal{S},\epsilon}_{\Omega,h}$$

be the connected component containing $x^{\mathcal{S},\epsilon}_{\tilde\pi}$. After possibly shrinking $\Omega \subset \mathscr{W}^Q_{\lambda_\pi}$, there exists an ideal $I_{\mathscr{C}^\epsilon} \subset \mathcal{O}_\Omega$ such that

$$\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\mathscr{C}} \cong \mathcal{O}_\Omega/I_{\mathscr{C}^\epsilon}.$$

Proof. By exactness of completion, and Proposition 8.12(i), the map $\widehat\Lambda \to \widehat{\mathbf{T}}^{\mathcal{S},\epsilon}_{\Omega,\tilde\pi}$ of completed (algebraic) local rings is surjective. As the completions are isomorphic we deduce

$$\widehat\Lambda_{\lambda_\pi} \twoheadrightarrow \widehat{\mathbf{T}}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}}.$$

As $\mathscr{E}^{\mathcal{S},\epsilon}_{\Omega,h}$ is finite over Ω, we may shrink Ω so that

$$w^{-1}(w(x^{\mathcal{S},\epsilon}_{\tilde\pi})) = \{x^{\mathcal{S},\epsilon}_{\tilde\pi}\},$$

so the weight map $w : \mathscr{C}^\epsilon \to \Omega$ is injective at $x^{\mathcal{S},\epsilon}_{\tilde\pi}$ in the sense of [27, §7.3.3, Prop. 4], and that proposition implies the natural map $\Lambda_{\lambda_\pi} \to \mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}}$ is surjective. We take $I_{x_{\tilde\pi}}$ to be the kernel of this map, proving (i). Part (ii) follows from (i) and the definition of rigid localisation, as in [11, Lem. 2.10]. ☐

To delocalise Proposition 8.12(ii), note there is a coherent sheaf $\mathscr{M}$ on $\mathscr{E}^{\mathcal{S},\epsilon}_{\Omega,h}$, with

$$\mathscr{M}(C) = \mathrm{H}^t_c(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h,\epsilon} \otimes_{\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,h}} \mathcal{O}_C,$$

given by the Hecke action on overconvergent cohomology. Its rigid localisation is

$$\begin{aligned}\mathscr{M}_{x_{\tilde\pi}} &= \varinjlim_{x^{\mathcal{S},\epsilon}_{\tilde\pi}\in C\subset\mathcal{E}^{\mathcal{S},\epsilon}_{\Omega,h}} \mathrm{H}^t_c(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h,\epsilon} \otimes_{\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,h}} \mathcal{O}_C \\ &\cong \mathrm{H}^t_c(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h,\epsilon} \otimes_{\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,h}} \mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}}.\end{aligned}$$

Proposition 8.14. *Let $\mathscr{C}^\epsilon$ be the connected component from Proposition 8.13. Perhaps after further shrinking Ω, we have $\mathrm{H}^t_c(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h,\epsilon} \otimes_{\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,h}} \mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\mathscr{C}}$ is free of rank 1 over $\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\mathscr{C}}$.*

Proof. We have

$$\begin{aligned}\mathscr{M}_{x_{\tilde\pi}} &\cong \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h,\epsilon}\otimes_{\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,h}}\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}}\\ &\cong \left[\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h,\epsilon}\otimes_{\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,h}}\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\tilde\pi}\right]\otimes_{\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\tilde\pi}}\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}}\\ &= \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\epsilon}_{\mathfrak{m}^{\mathcal{S}}_{\tilde\pi}}\otimes_{\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\tilde\pi}}\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}}.\end{aligned}$$

By Proposition 8.12(ii), this is free of rank 1 over $\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,x_{\tilde\pi}}$. We conclude again from [11, Lem. 2.10]. □

8.3.4. Étaleness of families. We now use Hypothesis 8.6.

Proposition 8.15. *(i) Let*

$$\mathscr{C}^\epsilon = \mathrm{Sp}(\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\mathscr{C}}) \subset \mathcal{E}^{\mathcal{S},\epsilon}_{\Omega,h}$$

be as in Proposition 8.14. If $\mathrm{Ev}^{\eta_0}_\beta(\Phi^\epsilon_{\tilde\pi}) \neq 0$ *for some* β*, then* $w : \mathscr{C}^\epsilon \to \Omega$ *is étale.*

(ii) If π *admits a non-zero Deligne-critical L-value at p with sign* ϵ*, then* $w : \mathscr{C}^\epsilon \to \Omega$ *is étale.*

Proof. Let $\Phi^\epsilon_{\mathscr{C}}$ be a generator of $\mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_\Omega)^{\leqslant h,\epsilon}\otimes_{\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,h}}\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\mathscr{C}}$ over $\mathbf{T}^{\mathcal{S},\epsilon}_{\Omega,\mathscr{C}}$, normalised so that $\mathrm{sp}_{\lambda_\pi}(\Phi^\epsilon_{\mathscr{C}}) = \Phi^\epsilon_{\tilde\pi}$. Combining Propositions 8.13 and 8.14, we have $\mathrm{Ann}_{\mathcal{O}_\Omega}(\Phi^\epsilon_{\mathscr{C}}) = I_{\mathscr{C}^\epsilon}$. Exactly as in Proposition 7.10, the non-vanishing hypothesis gives

$$0 = \mathrm{Ann}_{\mathcal{O}_\Omega}(\Phi^\epsilon_{\mathscr{C}}) = I_{\mathscr{C}^\epsilon},$$

giving (i). For (ii), we argue exactly as in Corollary 7.12; the sign condition (Definition 7.3) is now necessary due to the support of $\mathcal{E}^{j,\eta_0}_\chi$ (see Theorem 5.22). In (ii) we have used Hypothesis 8.6 for π. □

This proof is valid for any ϵ arising as the sign of a non-zero L-value, so if π admits any non-zero Deligne-critical L-value at p, then Proposition 8.15 holds for *some* ϵ. In Proposition 8.17, we will use Proposition 8.15 for one ϵ to deduce it for all ϵ.

Proposition 8.16. *Suppose* $\tilde\pi$ *is strongly non-Q-critical,* λ_π *is H-regular and* $\mathrm{Ev}^{\eta_0}_\beta(\Phi^{\epsilon_0}_{\tilde\pi}) \neq 0$ *for some* ϵ_0,β*. Then* $\mathscr{C}^{\epsilon_0}$ *contains a Zariski-dense set* $\mathscr{C}^{\epsilon_0}_{\mathrm{nc}}$ *of classical cuspidal non-Q-critical points.*

Proof. The conditions ensure $\mathscr{C}^{\epsilon_0} \to \Omega$ is étale, so $\dim(\mathscr{C}^{\epsilon_0}) = \dim(\Omega)$. We conclude exactly as in Proposition 7.14 using [19, Prop. 5.15]. □

Proposition 8.17. *Suppose* $\tilde\pi$ *is strongly non-Q-critical,* λ_π *is H-regular and that* $\mathrm{Ev}^{\eta_0}_\beta(\Phi^{\epsilon_0}_{\tilde\pi}) \neq 0$ *for some* ϵ_0 *and some* β*. Then* $\mathscr{C}^\epsilon$ *is independent of* $\epsilon \in \{\pm1\}^\Sigma$*, in the sense that for any such* ϵ*, there is a canonical isomorphism* $\mathscr{C}^{\epsilon_0} \xrightarrow{\sim} \mathscr{C}^\epsilon$ *over* Ω*.*

Proof. By Propositions 8.15 and 8.16, $\mathscr{C}^{\epsilon_0}$ is étale over Ω and $\mathscr{C}^{\epsilon_0}_{\mathrm{nc}} \subset \mathscr{C}^{\epsilon_0}$ is Zariski-dense. Now let ϵ be arbitrary. At any $y^{\epsilon_0} \in \mathscr{C}^{\epsilon_0}_{\mathrm{nc}}$ of weight λ_y corresponding to $\tilde\pi_y$, by Proposition 2.3 and non-Q-criticality we have

$$0 \neq \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{V}^\vee_{\lambda_y})^\epsilon_{\mathfrak{m}^{\mathcal{S}}_y} \cong \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)},\mathscr{D}_{\lambda_y})^\epsilon_{\mathfrak{m}^{\mathcal{S}}_y}.$$

By Remark 7.9, since y^{ϵ_0} is cuspidal there exists $y^\epsilon \in \mathscr{E}^{\mathcal{S},\epsilon}_{\Omega,h}$ corresponding to $\tilde\pi_y$. As in Corollary 7.22, the map

$$\mathscr{C}^{\epsilon_0}_{\mathrm{nc}} \to \mathscr{E}^{\mathcal{S},\epsilon}_{\Omega,h}, \qquad y^{\epsilon_0} \mapsto y^\epsilon$$

interpolates to a closed immersion

$$\iota^\epsilon : \mathscr{C}^{\epsilon_0} \hookrightarrow \mathscr{E}_{\Omega,h}^{\mathcal{S},\epsilon}$$

sending $x_{\tilde\pi}^{\mathcal{S},\epsilon_0}$ to $x_{\tilde\pi}^{\mathcal{S},\epsilon}$. Thus $\iota^\epsilon(\mathscr{C}^{\epsilon_0}) \subset \mathscr{C}^\epsilon$, so $\mathscr{C}^\epsilon$ contains an irreducible component of dimension $\dim(\Omega)$. As $\mathcal{O}_{\mathscr{C}^\epsilon} \cong \mathcal{O}_\Omega/I_{\mathscr{C}^\epsilon}$, we deduce $I_{\mathscr{C}^\epsilon} = 0$, so $\mathscr{C}^\epsilon \to \Omega$ is étale at $\tilde\pi$, and conclude that ι^ϵ is an isomorphism as in Corollary 7.22. □

Hence $\mathscr{E}_{\Omega,h}^{\mathcal{S}}$ and all the $\mathscr{E}_{\Omega,h}^{\mathcal{S},\epsilon}$ are locally isomorphic at $\tilde\pi$, so we drop ϵ from notation. We deduce existence and étaleness of the component $\mathscr{C} \subset \mathscr{E}_{\Omega,h}^{\mathcal{S}}$ in Theorem 8.11; we take it to be any of the $\mathscr{C}^\epsilon$. The isomorphisms between the $\mathscr{C}^\epsilon$ identify all the $x_{\tilde\pi}^{\mathcal{S},\epsilon}$ with a single point $x_{\tilde\pi}^{\mathcal{S}} \in \mathscr{E}_{\Omega,h}^{\mathcal{S}}$.

Remark 8.18. Having a family of *cuspidal* automorphic representations is essential here; e.g. for GL_2, an Eisenstein series will appear in only one of the $\pm$-eigencurves (see [20, §3.2.6]).

8.3.5. Eigenclasses for Shalika families. We now refine Proposition 8.16. Suppose $\tilde\pi$ satisfying (C1-2) of Conditions 2.8 is strongly non-Q-critical. For $v \in S$, assume π_v admits a Shalika new vector of conductor $c(\pi_v)$ and Hypothesis 8.6 holds for $c = c(\pi_v)$. Suppose λ_π is H-regular and π admits a non-zero Deligne-critical L-value at p. Then by §8.3.4, we know:

(1) that there is a unique irreducible component $\mathscr{C}$ of $\mathscr{E}_{\Omega,h}^{\mathcal{S}}$ through $x_{\tilde\pi}^{\mathcal{S}}$;

(2) that $\mathscr{C} = \mathrm{Sp}(\mathbf{T}_{\Omega,\mathscr{C}}^{\mathcal{S}}) = \mathrm{Sp}(\mathbf{T}_{\Omega,\mathscr{C}}^{\mathcal{S},\epsilon})$ for all ϵ; and

(3) that $w : \mathscr{C} \to \Omega$ is étale.

As in Proposition 7.14, we deduce that $\mathscr{C}$ contains a Zariski-dense set $\mathscr{C}_{\mathrm{nc}}$ of classical cuspidal non-Q-critical points. If $y \in \mathscr{C}_{\mathrm{nc}}$ it corresponds to a Q-refined RACAR $\tilde\pi_y$, since by construction y appears in cohomology at (parahoric-at-p) level $K(\tilde\pi)$.

Proposition 8.19. *For each $\epsilon \in \{\pm1\}^\Sigma$, up to shrinking Ω, there exists a Hecke eigenclass $\Phi_{\mathscr{C}}^\epsilon \in \mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_\Omega)^\epsilon$ such that for each $y \in \mathscr{C}_{\mathrm{nc}}$:*

(i) $\mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_{\lambda_y})_{\mathfrak{m}_y^{\mathcal{S}}}^\epsilon$ is a line generated by $\mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^\epsilon)$, where $\lambda_y := w(y)$,

(ii) the Q-refined RACAR $\tilde\pi_y$ satisfies (C1),

(iii) for all $v \in S$, $\pi_{y,v}$ admits a Shalika new vector of conductor $c(\pi_{y,v}) = c(\pi_v)$, and

(iv) $\tilde\pi_y$ satisfies (C2).

Proof. (i) By Propositions 8.15 and 8.17, $\mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_\Omega)^{\leqslant h,\epsilon} \otimes_{\mathbf{T}_{\Omega,h}^{\mathcal{S},\epsilon}} \mathbf{T}_{\Omega,\mathscr{C}}^{\mathcal{S},\epsilon} \subset \mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_\Omega)^\epsilon$ is free of rank one over $\mathcal{O}_\Omega$; let $\Phi_{\mathscr{C}}^\epsilon$ be any generator. The Hecke operators act by scalars on this $\mathcal{O}_\Omega$-line, so $\Phi_{\mathscr{C}}^\epsilon$ is a Hecke eigenclass. By Proposition 8.15 the structure map $\mathcal{O}_\Omega \to \mathbf{T}_{\Omega,\mathscr{C}}^{\mathcal{S},\epsilon}$ is an isomorphism, mapping $\mathfrak{m}_{\lambda_y}$ bijectively to $\mathfrak{m}_y^{\mathcal{S}}$. Thus specialisation at λ_y, induced from reduction modulo $\mathfrak{m}_{\lambda_y}$ as an $\mathcal{O}_\Omega$-module, is equivalent to reduction modulo $\mathfrak{m}_y^{\mathcal{S}}$ as a $\mathbf{T}_{\Omega,\mathscr{C}}^{\mathcal{S},\epsilon}$-module; whence

$$\begin{aligned}\mathrm{sp}_{\lambda_y} : \mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_\Omega)^{\leqslant h,\epsilon} \otimes_{\mathbf{T}_{\Omega,h}^{\mathcal{S},\epsilon}} \mathbf{T}_{\Omega,\mathscr{C}}^{\mathcal{S},\epsilon} &\twoheadrightarrow \mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_\Omega)^{\leqslant h,\epsilon} \otimes_{\mathbf{T}_{\Omega,h}^{\mathcal{S},\epsilon}} \mathbf{T}_{\Omega,\mathscr{C}}^{\mathcal{S},\epsilon}/\mathfrak{m}_y^{\mathcal{S}} \\ &\cong \mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_\Omega)_{\mathfrak{m}_y^{\mathcal{S}}}^\epsilon \otimes_{\mathbf{T}_{\Omega,\tilde\pi_y}^{\mathcal{S},\epsilon}} (\mathbf{T}_{\Omega,\tilde\pi_y}^{\mathcal{S},\epsilon}/\mathfrak{m}_y^{\mathcal{S}}) \\ &\cong \mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_\Omega)_{\mathfrak{m}_y^{\mathcal{S}}}^\epsilon \otimes_{\Lambda_y} (\Lambda_y/\mathfrak{m}_{\lambda_y}) \cong \mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_{\lambda_y})_{\mathfrak{m}_y^{\mathcal{S}}}^\epsilon, \qquad (8.5)\end{aligned}$$

where $\Lambda_y := \mathcal{O}_{\Omega,\mathfrak{m}_{\lambda_y}}$. Here the first isomorphism follows since reduction at $\mathfrak{m}_y^{\mathcal{S}}$ factors through localisation at $\mathfrak{m}_y^{\mathcal{S}}$, the second as the weight map is an isomorphism at y, and the third by Remark 7.9. Hence $\mathrm{H}_{\mathrm{c}}^t(S_{K(\tilde\pi)}, \mathscr{D}_{\lambda_y})_{\mathfrak{m}_y^{\mathcal{S}}}^\epsilon$ is a line over L, generated by $\mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^\epsilon)$, which proves (i).

(ii) To prove each $y \in \mathscr{C}_{\mathrm{nc}}$ satisfies (C1), we argue exactly as in Proposition 7.16; here we already have étaleness, so this case is easier, and we are terse with details. Fix (χ, j) with

$L^{(p)}(\pi\otimes\chi, j+\frac{1}{2}) \neq 0$ (by hypothesis), where χ has conductor p^β, with $\beta_{\mathfrak{p}} \geqslant 1$ for all $\mathfrak{p}|p$. Let $\epsilon = (\chi\chi_{\mathrm{cyc}}^j\eta)_\infty$, and define a map

$$\mathrm{Ev}_{\chi,j}^{\Omega} : \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)}, \mathscr{D}_\Omega) \to \mathcal{O}_\Omega$$

such that (up to shrinking Ω) $\mathrm{Ev}_{\chi,j}^{\Omega}(\Phi_{\mathscr{C}}^\epsilon)$ is everywhere non-vanishing on Ω. Recall $\mathscr{E}_\chi^{j,\eta_0}$ from (5.4). Then for all $y \in \mathscr{C}_{\mathrm{nc}}$, we have

$$\begin{aligned}\mathrm{Ev}_{\chi,j}^{\Omega}(\Phi_{\mathscr{C}}^\epsilon)(\lambda_y) &= \int_{\mathrm{Gal}_p} \chi\ \chi_{\mathrm{cyc}}^j \cdot \mathrm{Ev}_\beta^{\eta_0}(\mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^\epsilon)) \\ &= \mathcal{E}_\chi^{j,\eta_0}\left(r_{\lambda_y}\circ \mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^\epsilon)\right) \neq 0.\end{aligned} \tag{8.6}$$

As (8.6) is non-zero, each π_y satisfies (C1) by Proposition 5.15(cf. Proposition 7.16), showing (ii).

We need the following in proving both (iii) and (iv). Combining (i) with non-Q-criticality shows

$$\dim_L \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)}, \mathscr{V}_{\lambda_y}^\vee(L))^\epsilon_{\mathfrak{m}_y^{\mathcal{S}}} = 1.$$

Base-changing, the same is true with $\overline{\mathbf{Q}}_p$-coefficients. By Proposition 2.3, we see:

($\dagger$) the generalised $\mathcal{H}^{\mathcal{S}}$-eigenspace in $\pi_{y,f}^{K(\tilde\pi)}$ at $\mathfrak{m}_y^{\mathcal{S}}$ is a line over $\mathbf{C}$.

(iii) We now study π_y at $v\in S$. Letting $\eta_y = \eta_0|\cdot|^{\mathsf{w}_y}$, by ($\dagger$) we have

$$\dim_{\mathbf{C}}\ (\pi_{y,v})^{J_v^{\eta_y}(c)}\big[S_{\alpha_v} - \eta_{y,v}(\alpha_v) : \alpha_v \in \mathcal{O}_v^\times\big]\ = 1.$$

Lemma 8.4 implies that $\pi_{y,v}$ admits a Shalika new vector of conductor $c = c(\pi_v)$, giving (iii).

(iv) As in (iii), $\pi_{\mathfrak{p},y}$ is parahoric-spherical as ($\dagger$) implies

$$\dim_{\mathbf{C}}\ \pi_{\mathfrak{p},y}^{J_{\mathfrak{p}}}\big[U_{\mathfrak{p}}^\circ - \alpha_{\mathfrak{p},y}^\circ\big] = \dim_{\mathbf{C}}\ \mathcal{S}_{\psi_{\mathfrak{p}}}^{\eta_{\mathfrak{p},y}}(\pi_{\mathfrak{p},y}^{J_{\mathfrak{p}}})\big[U_{\mathfrak{p}}^\circ - \alpha_{\mathfrak{p},y}^\circ\big]\ = 1, \tag{8.7}$$

where $\alpha_{\mathfrak{p},y}^\circ$ is the $U_{\mathfrak{p}}^\circ$-eigenvalue of $\tilde\pi_y$. It remains to show the non-vanishing in (C2).

Let $W_{\mathfrak{p},y}$ be a generator of (8.7) for $\mathfrak{p}|p$. Using (C1) at $v\notin S\cup\{\mathfrak{p}|p\}$, and Hypothesis 8.6 and the equality $c(\pi_{y,v}) = c(\pi_v)$ for $v\in S$, we may take Friedberg–Jacquet test vectors $W_{y,v}^{\mathrm{FJ}}$ for $v\nmid p$ such that

$$W_{y,f}^{\mathrm{FJ}} = \otimes_{v\nmid\mathfrak{p}} W_{y,v}^{\mathrm{FJ}} \otimes_{\mathfrak{p}|p} W_{\mathfrak{p},y}\ \in \mathcal{S}_{\psi_f}^{\eta_{y,f}}\left(\pi_{y,f}^{K(\tilde\pi)}\right)$$

is fixed by $K(\tilde\pi)$. For ϵ as in (8.6), let

$$\phi_y^\epsilon := \Theta_{i_p}^{K(\tilde\pi),\epsilon}(W_{y,f}^{\mathrm{FJ}}) \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde\pi)}, \mathscr{V}_{\lambda_y}^\vee(\overline{\mathbf{Q}}_p))^\epsilon_{\mathfrak{m}_y^{\mathcal{S}}}.$$

This line contains $r_{\lambda_y}\circ\mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^\epsilon)$, so there is $c_y^\epsilon \in \overline{\mathbf{Q}}_p^\times$ such that $c_y^\epsilon\phi_y^\epsilon = r_{\lambda_y}\circ\mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^\epsilon)$. Then

$$c_y^\epsilon\cdot\mathcal{E}_\chi^{j,\eta_0}\left(\phi_y^\epsilon\right) = \mathcal{E}_\chi^{j,\eta_0}\left(r_{\lambda_y}\circ\mathrm{sp}_{\lambda_y}(\Phi_{\mathscr{C}}^\epsilon)\right) \neq 0,$$

where non-vanishing is (8.6). As in Theorem 5.22 (via Lemma 5.16 and Proposition 5.20), the left-hand side is

$$\begin{aligned}&\Bigg[c_y^\epsilon\gamma_{p\mathfrak{m}}\lambda_y(t_p^\beta)\mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d})^{jn}\tau(\chi_f)^n \\ &\qquad\times\textstyle\prod_{\mathfrak{p}|p} e_{\mathfrak{p}}(\tilde\pi_y,\chi,j)e_\infty(\pi_y,\chi,j)L^{(p)}(\pi_y\otimes\chi, j+\frac{1}{2})\Bigg]\cdot\prod_{\mathfrak{p}|p} W_{y,\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}) \neq 0.\end{aligned}$$

As the L-function is analytic, the bracketed term is finite scalar; we deduce that each $W_{y,\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}) \neq 0$. We can renormalise $W_{y,\mathfrak{p}}$ (and hence c_y^ϵ) so that $W_{y,\mathfrak{p}}(t_{\mathfrak{p}}^{-\delta_{\mathfrak{p}}}) = 1$, so (C2) holds. □

With this, we have completed the proof of Theorem 8.11.

8.4. Families of p-adic L-functions. Let $\tilde{\pi}$ satisfy (C1-2) of Conditions 2.8. We also assume that all the hypotheses of Theorem 8.11 are satisfied.

Let $\mathscr{C} \subset \mathscr{E}^{\mathcal{S}}_{\Omega,h}$ be the unique (Shalika) family through $x^{\mathcal{S}}_{\tilde{\pi}}$, and $\mathscr{C}_{\mathrm{nc}}$ the Zariski-dense subset of classical points, both from Theorem 8.11. For each $\epsilon \in \{\pm 1\}^{\Sigma}$ let $\Phi^{\epsilon}_{\mathscr{C}} \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_{\Omega})^{\epsilon}$ be the resulting Hecke eigenclass. We may renormalise $\Phi^{\epsilon}_{\mathscr{C}}$ so that $\mathrm{sp}_{\lambda}(\Phi^{\epsilon}_{\mathscr{C}}) = \Phi^{\epsilon}_{\tilde{\pi}}$. The following is an analogue of Definition 6.22 for families.

Definition 8.20. Let $\mathcal{L}^{\mathscr{C},\epsilon}_p := \mu^{\eta_0}(\Phi^{\epsilon}_{\mathscr{C}})$. Also let

$$\Phi_{\mathscr{C}} = \sum_{\epsilon} \Phi^{\epsilon}_{\mathscr{C}} \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_{\Omega}),$$

which is also a Hecke eigenclass. Define the *p-adic L-function over* $\mathscr{C}$ to be

$$\mathcal{L}^{\mathscr{C}}_p := \mu^{\eta_0}(\Phi_{\mathscr{C}}) = \sum_{\epsilon \in \{\pm 1\}^{\Sigma}} \mathcal{L}^{\mathscr{C},\epsilon}_p \in \mathcal{D}(\mathrm{Gal}_p, \mathcal{O}_{\Omega}).$$

Via the Amice transform, as in Definition 6.22, after identifying $\mathscr{C}$ with Ω via w we may consider $\mathcal{L}^{\mathscr{C}}_p$ as a rigid function $\mathscr{C} \times \mathscr{X}(\mathrm{Gal}_p) \to \overline{\mathbf{Q}}_p$.

The following implies Theorem C of the introduction. The hard/novel part of the proof has already been handled; given Theorem 8.11, the remainder is standard.

Theorem 8.21. *Suppose $\tilde{\pi}$ satisfies the hypotheses of Theorem 8.11. Let $y \in \mathscr{C}_{\mathrm{nc}}$ be a classical cuspidal point attached to a non-Q-critical Q-refined RACAR $\tilde{\pi}_y$ satisfying (C1-2). For each ϵ, there exists a p-adic period $c^{\epsilon}_y \in L^{\times}$ such that*

$$\mathcal{L}^{\mathscr{C},\epsilon}_p(y,-) = c^{\epsilon}_y \cdot \mathcal{L}^{\epsilon}_p(\tilde{\pi}_y,-) \tag{8.8}$$

as functions $\mathscr{X}(\mathrm{Gal}_p) \to \overline{\mathbf{Q}}_p$. In particular, $\mathcal{L}^{\mathscr{C}}_p$ satisfies the following interpolation: for any $j \in \mathrm{Crit}(w(y))$, and for any Hecke character χ of conductor p^{β} with $\beta_{\mathfrak{p}} > 1$ for all $\mathfrak{p}|p$, we have

$$i_p^{-1}(\mathcal{L}^{\mathscr{C}}_p(y, \chi\chi^j_{\mathrm{cyc}})) = c^{\epsilon}_y A \tau(\chi_f)^n \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d})^{jn} \prod_{\mathfrak{p}|p} e_{\mathfrak{p}}(\tilde{\pi}_y, \chi, j) e_{\infty}(\pi_y, \chi, j) \frac{L^{(p)}(\pi_y \times \chi, j+1/2)}{\Omega^{\epsilon}_{\pi_y}},$$

where $\epsilon = (\chi\chi^j_{\mathrm{cyc}}\eta)_{\infty}$ and other notation is as in Theorem 6.23. Finally $c^{\epsilon}_{x^{\mathcal{S}}_{\tilde{\pi}}} = 1$ for all ϵ.

Remark 8.22. The complex periods $\Omega^{\epsilon}_{\pi_y}$ are only well-defined up to multiplication by $E^{\times}$, where E is the number field from Definition 2.9; the numbers c^{ϵ}_y are p-adic analogues.

Proof. Let y be as in the theorem and put $\lambda_y = w(y)$. As in §2.8 (using Hypothesis 8.6), fix a Friedberg–Jacquet test vector

$$W^{\mathrm{FJ}}_{y,f} \in \mathcal{S}^{\eta_{y,f}}_{\psi_f}(\pi_{y,f})^{K(\tilde{\pi})},$$

and for each ϵ a complex period $\Omega^{\epsilon}_{\pi_y}$ as in §2.10. Since $y \in \mathscr{C}$ is defined over L, as in §2.10 there exists a class

$$\phi^{\epsilon}_y := \Theta^{K,\epsilon}_{i_p}(W^{\mathrm{FJ}}_{y,f})/i_p(\Omega^{\epsilon}_{\pi_y}) \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{V}^{\vee}_{\lambda_y}(L))^{\epsilon}_{\mathfrak{m}^{\mathcal{S}}_y}.$$

Via non-Q-criticality, we lift ϕ^{ϵ}_y to a non-zero class $\Phi^{\epsilon}_y \in \mathrm{H}^t_{\mathrm{c}}(S_{K(\tilde{\pi})}, \mathscr{D}_{\lambda_y}(L))^{\epsilon}_{\mathfrak{m}^{\mathcal{S}}_y}$. By Theorem 8.11, this space is equal to $L \cdot \mathrm{sp}_{\lambda_y}(\Phi^{\epsilon}_{\mathscr{C}})$, so there exists $c^{\epsilon}_y \in L^{\times}$ such that

$$\mathrm{sp}_{\lambda_y}(\Phi^{\epsilon}_{\mathscr{C}}) = c^{\epsilon}_y \cdot \Phi^{\epsilon}_y.$$

By definition, $\mathcal{L}^{\epsilon}_p(\tilde{\pi}_y) = \mu^{\eta_0}(\Phi^{\epsilon}_y)$. As evaluation maps commute with weight specialisation (Proposition 6.15), we find

$$\mathrm{sp}_{\lambda_y}(\mathcal{L}^{\mathscr{C},\epsilon}_p) = c^{\epsilon}_y \cdot \mathcal{L}^{\epsilon}_p(\tilde{\pi}_y),$$

which is a reformulation of (8.8). The interpolation formula then follows from Remark 6.24. Finally, our normalisation of $\Phi^{\epsilon}_{\mathscr{C}}$ ensures $c^{\epsilon}_{x^{\mathcal{S}}_{\tilde{\pi}}} = 1$. □

8.5. Unicity of non-Q-critical p-adic L-functions. In Proposition 6.25, we proved a unicity result for $\mathcal{L}_p(\tilde{\pi})$ when $\tilde{\pi}$ has sufficiently small slope. Whilst non-Q-critical slope implies non-Q-critical, the converse is false; even when $G = \mathrm{GL}_2$, there exist critical slope refinements that are non-critical (e.g. [69, 20]). We now use $\mathcal{L}_p^{\mathscr{C}}$ to prove an analogue of Proposition 6.25 for the wider class of (non-Q-critical) $\tilde{\pi}$ satisfying the hypotheses of Theorem 8.11.

Let $\alpha_p^\circ = \prod_{\mathfrak{p}|p}(\alpha_{\mathfrak{p}}^\circ)^{e_{\mathfrak{p}}}$ and $h_p := v_p(\alpha_p^\circ)$. For $\epsilon \in \{\pm 1\}^\Sigma$, let $\mathscr{X}(\mathrm{Gal}_p)^\epsilon$ be the component of characters χ with $\epsilon = (\chi\eta)_\infty$. Then

$$\mathscr{X}(\mathrm{Gal}_p) = \bigsqcup_\epsilon \mathscr{X}(\mathrm{Gal}_p)^\epsilon$$

(e.g. [23, Rem. 7.3.4]). If $\mathcal{L} : \mathscr{C} \times \mathscr{X}(\mathrm{Gal}_p) \to L$ is a rigid analytic function, then $\mathcal{L} = \sum_\epsilon \mathcal{L}^\epsilon$, with $\mathcal{L}^\epsilon$ supported on $\mathscr{C} \times \mathscr{X}(\mathrm{Gal}_p)^\epsilon$.

Proposition 8.23. *Suppose $\tilde{\pi}$ satisfies the hypotheses of Theorem 8.11. Suppose Leopoldt's conjecture holds for F at p and that $\mathcal{L}_p^\epsilon(\tilde{\pi}) \neq 0$. Let*

$$\mathcal{L}^\epsilon : \mathscr{C} \times \mathscr{X}(\mathrm{Gal}_p)^\epsilon \to L$$

be any rigid analytic function such that for all $y \in \mathscr{C}_{\mathrm{nc}}$, the specialisation $\mathcal{L}^\epsilon(y, -)$ is admissible of growth h_p and there exists $C_y^\epsilon \in L^\times$ such that

$$i_p^{-1}(\mathcal{L}_p^{\mathscr{C}}(y, \chi\chi_{\mathrm{cyc}}^j)) = C_y^\epsilon A\tau(\chi_f)^n \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d})^{jn} \prod_{\mathfrak{p}|p} e_{\mathfrak{p}}(\tilde{\pi}_y, \chi, j) \cdot e_\infty(\pi_y, \chi, j) \frac{L^{(p)}(\pi_y \times \chi, j+1/2)}{\Omega_{\pi_y}^\epsilon}, \quad (8.9)$$

for all finite order $\chi \in \mathscr{X}(\mathrm{Gal}_p)$ and $j \in \mathrm{Crit}(\lambda_y)$ such that $(\chi\chi_{\mathrm{cyc}}^j\eta)_\infty = \epsilon$. Then there exists $C \in L$ such that

$$\mathcal{L}^\epsilon\big(x_{\tilde{\pi}}^{\mathcal{S}}, -\big) = C \cdot \mathcal{L}_p^\epsilon(\tilde{\pi}) \in \mathcal{D}(\mathrm{Gal}_p, L).$$

Proof. Let

$$\mathcal{B}^\epsilon = \mathcal{L}^\epsilon / \mathcal{L}_p^{\mathscr{C},\epsilon} \in \mathrm{Frac}\big(\mathcal{O}(\mathscr{C} \times \mathscr{X}(\mathrm{Gal}_p)^\epsilon)\big).$$

We claim $\mathcal{L}_p^{\mathscr{C},\epsilon}$ is not a zero-divisor, so this is well-defined. Note $\mathscr{C}$ contains a Zariski-dense set of classical points y of regular weight; at every such point, let $j_y = \max(\mathrm{Crit}(\lambda_y))$. Fix such a y. For any everywhere ramified finite order Hecke character $\chi \in \mathscr{X}(\mathrm{Gal}_p)$ with $\chi(z)z^{j_y} \in \mathscr{X}(\mathrm{Gal}_p)^\epsilon$, we have

$$\mathcal{L}_p^{\mathscr{C},\epsilon}(y, \chi(z)z^{j_y}) = c_y^\epsilon A\tau(\chi_f)^n \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{d})^{jn} \prod_{\mathfrak{p}|p} e_{\mathfrak{p}}(\tilde{\pi}_y, \chi, j) e_\infty(\pi_y, \chi, j) \frac{L^{(p)}(\pi_y \times \chi, j+1/2)}{\Omega_{\pi_y}^\epsilon} \neq 0. \quad (8.10)$$

Every connected component of $\mathscr{X}(\mathrm{Gal}_p)^\epsilon$ contains such a character $\chi(z)z^{j_y}$, and on every such component the only zero-divisor is 0; we conclude that $\mathcal{L}_p^{\mathscr{C},\epsilon}(y, -)$ is not a zero-divisor. Now suppose $\mathcal{M}\mathcal{L}_p^{\mathscr{C},\epsilon} = 0$ for some $\mathcal{M} \in \mathcal{O}(\mathscr{C} \times \mathscr{X}(\mathrm{Gal}_p)^\epsilon)$. Then $\mathcal{M}(y, -) = 0$ for all y as above. Since this is true for a Zariski-dense set, we have $\mathcal{M} = 0$, and $\mathcal{L}_p^{\mathscr{C},\epsilon}$ is not a zero-divisor.

For a Zariski-dense subset of classical $y \in \mathscr{C}$, we have $h_p < \#\mathrm{Crit}(\lambda_y)$. Fix such a y. Since the slope at p is constant in p-adic families, y has slope h_p and hence both $\mathcal{L}^\epsilon(y, -)$ and $\mathcal{L}_p^{\mathscr{C},\epsilon}(y, -)$ are admissible of growth h_p. Thus by (8.9) and Proposition 6.25 we know

$$\mathcal{B}^\epsilon(y, -) = C_y^\epsilon / c_y^\epsilon \in L^\times$$

is constant (as a function on $\mathscr{X}(\mathrm{Gal}_p)^\epsilon$). By Zariski-density of such points y, we deduce that $\mathcal{B}^\epsilon(y, -) \in \mathbf{P}^1(L)$ is constant for each $y \in \mathscr{C}$. Moreover $\mathcal{B}^\epsilon(y, -)$ does not have a pole at $y = x_{\tilde{\pi}}^{\mathcal{S}}$ since

$$\mathcal{L}_p^{\mathscr{C},\epsilon}(x_{\tilde{\pi}}^{\mathcal{S}}, -) = \mathcal{L}_p^\epsilon(\tilde{\pi}) \neq 0$$

by assumption, so $\mathcal{B}^\epsilon(x^{\mathcal{S}}_{\tilde\pi}, -) = C$ for some $C \in L$. In particular

$$\mathcal{L}^\epsilon(x^{\mathcal{S}}_{\tilde\pi}, -) = C\mathcal{L}_p^{\mathscr{C},\epsilon}(x^{\mathcal{S}}_{\tilde\pi}, -) = C\mathcal{L}_p^\epsilon(\tilde\pi). \qquad \square$$

The following is a reformulation of Proposition 8.23:

Corollary 8.24. *Suppose $\tilde\pi$ satisfies the hypotheses of Theorem 8.11. Assume Leopoldt's conjecture for F at p. Up to scaling the p-adic periods, $\mathcal{L}_p(\tilde\pi)$ is uniquely determined by interpolation of L-values over the unique Shalika family $\mathscr{C}$ of level $K(\tilde\pi)$ through $\tilde\pi$.*

In particular, up to these assumptions $\mathcal{L}_p(\tilde\pi)$ does not depend on our method of construction.

Remarks 8.25. We expect that $\mathcal{L}_p^\epsilon(\tilde\pi)$ should always be non-zero. By (8.9) and Lemma 7.4, if λ_π is regular this is automatic for any ϵ such that there exists a finite order Hecke character χ such that $(\chi_{\mathrm{cyc}}^j\chi\eta)_\infty = \epsilon$, where j is any integer strictly above the centre of $\mathrm{Crit}(\lambda_\pi)$.

Without Leopoldt, there is still an analogue for the restriction to 1-dimensional slices of Gal_p (cf. [11, Thm. 4.7(ii)] or §6.6). Thus the restriction of $\mathcal{L}_p(\tilde\pi)$ to the cyclotomic line is unique.

A. Errata for earlier works

Whilst writing this paper, we found errors in our earlier publications. We clarify them here.

(1) In [19, Rem. 4.19], which compared the right actions used in that paper with the left actions used in this, in the final sentence U_p^* should have been $\lambda(\sigma(t)^{-1}t)U_p^\cdot$ (not $\lambda^\vee(\sigma(t)^{-1}t)U_p^\cdot$). This was not used elsewhere *ibid.*; we have used the correct formulation here.

(2) The power of q in the statement of [38, Prop. 3.4] is incorrect. In the proof, one can reduce the support of the integral in the penultimate displayed equation to the Iwahori subgroup, not to $N_n^-(\mathcal{P}^\beta)T_n(\mathcal{O})N_n(\mathcal{P}^\beta)$ as stated, so the final volume term is wrong. The proof otherwise holds. A corrected statement is Proposition 5.20 of the present paper. (This ensures the final interpolation result is consistent with the Coates–Perrin-Riou conjecture on existence of p-adic L-functions; see [7, §3]. Indeed, [38, Thm. B] is not consistent with Coates–Perrin-Riou). The powers of q in Theorem B and Theorem 4.7 of [38] are thus incorrect. The interpolation formulas there should be replaced by that of Theorem A here.

Glossary of key notation/terminology

$\mathcal{A} = \mathcal{A}^Q$ ...... Locally analytic function space (§3.2)
$\alpha_{\mathfrak{p}}, \alpha_p$ .... $U_{\mathfrak{p}}, U_p$ eigenvalues (§2.7)
$\alpha_{\mathfrak{p}}^\circ, \alpha_p^\circ$ .... $U_{\mathfrak{p}}^\circ, U_p^\circ$ eigenvalues (§3.3)
$\beta = (\beta_{\mathfrak{p}})_{\mathfrak{p}|p} \in \mathbf{Z}^{\mathfrak{p}|p}$ .... Multi-index (§4.1)
(C1),(C2) .... Assumptions on $\tilde\pi$ (Cond. 2.8)
$\mathscr{C}$ .... Connected component of $\mathscr{E}_{\Omega,h}$ (§7.3)
$\mathscr{C}\ell_F^+(I)$ .... Narrow ray class gp. cond. I (§2.1)
$\mathrm{Crit}(\lambda)$ ... Deligne-critical L-values for λ (eqn. (2.2))
$\mathcal{D}$ .... Locally analytic distributions (§6.1.1)
$\mathcal{D}_\lambda = \mathcal{D}_\lambda^Q$ .... Q-parahoric dists. of wt. λ (§3.2.2)
$\mathcal{D}_\Omega = \mathcal{D}_\Omega^Q$ .... Q-parahoric dists. over Ω (§3.2.3)
$\mathscr{D}$ .... Local system of distributions (§2.3.2)
Δ_p .... Monoid in $G(\mathbf{Q}_p)$ gen. by $J_p, t_{\mathfrak{p}}$ (§3.3)
δ .... Representative of $\pi_0(X_\beta)$ (§4.1)
$\mathrm{Ev}_{\beta,\delta}^M$ .... Abstract evaluation map (Def. 4.5)
$\mathrm{Ev}_\beta^{\eta_0}$ .... Galois evaluation (eqn. (6.15))
$\mathcal{E}_\chi^{j,\eta_0}$ ... Classical evaluation map at χ, j (eqn. (5.4))
$\mathscr{E}_{\Omega,h}$ ...... Q-parabolic eigenvariety for G (Def. 7.1)
$\mathscr{E}_{\Omega,h}^{\mathcal{S},\epsilon}$ ...... 'modified' Q-par. eigenvariety (Def. 8.10)
$e_{\mathfrak{p}}(\tilde\pi,\chi,j)$ .... C–PR factor at p (Thm. 6.23)
$e'_{\mathfrak{p}}(\tilde\pi,\chi,j)$ .... Def. 5.19
$e_\infty(\pi,\chi,j)$ .... C–PR factor at ∞ (Def. 5.18
ϵ .... Character $K_\infty/K_\infty^\circ \to \{\pm1\}^\Sigma$ (§2.3.4)
F .... Totally real field of degree d
$\phi_{\tilde\pi}^\epsilon$ .... Classical class in $\mathrm{H}_{\mathrm{c}}^t$ (Def. 2.10)
$\Phi_{\tilde\pi}^\epsilon, \Phi_{\tilde\pi}$ .... Overconvergent classes in $\mathrm{H}_{\mathrm{c}}^t$ (§6.6)
G .... $\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\mathrm{GL}_{2n}$
G_n .... $\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\mathrm{GL}_n$
Gal_p .... $\mathrm{Gal}(F^{p\infty}/F)$ (§2.1)

$\mathrm{Gal}_p^{\mathrm{cyc}}$ ... $\mathrm{Gal}(\mathbf{Q}^{p\infty}/\mathbf{Q})$ (§2.1)
$\Gamma_{\beta,\delta}$ Arithmetic group in automorphic cycle (§4.2.2)
H ... $\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}[\mathrm{GL}_n \times \mathrm{GL}_n]$
$\mathcal{H}', \mathcal{H}$ ... Universal Hecke algebras (§2.4, Def. 2.9)
$\mathcal{H}^S$. Universal Hecke algebra at all primes (Def. 8.7)
$\leqslant h$ ... Slope $U_p^\circ \leqslant h$ part (§3.3)
$\mathscr{I}$ ... Irreducible component of $\mathscr{E}_{\Omega,h}$
i_p ... Fixed isomorphism $\mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$
ι ... Map $H \hookrightarrow G, (h_1,h_2) \mapsto \mathrm{diag}(h_1,h_2)$
ι_β ... Map in automorphic cycle (eqn. (4.3))
η ... Shalika character (§2.6)
J_p ... Parahoric subgroup of type Q (§2.7)
J_p^β ... Eqn. (6.8)
K ... Open cpct. subgp. of $G(\mathbf{A}_f)$ (eqn. (2.20))
$K(\tilde{\pi})$ ... Friedberg–Jacquet level (eqn. (2.23)),§8.3.1)
$K_1(\tilde{\pi})$ ... Whittaker new level (eqn. (7.2))
$\kappa_{\lambda,j}$ ... Map $V_\lambda^\vee \to V_{(j,-\mathsf{w}-j)}^H$ (§5.2)
$\kappa_{\lambda,j}^\circ$ ... Normalised map $V_\lambda^\vee(L) \to L$ (Def. 5.10)
L ... Extension of $\mathbf{Q}_p$, coefficient field (§2.10)
L_β ... Open compact in $H(\mathbf{A}_f)$ (Def. 4.2)
$L(\pi,s)$ ... Standard L-fn. of π
$L^{(p)}(\pi,s)$ ... $L()$ with no Euler factors at p
$\mathcal{L}_p(\tilde{\pi})$ ... p-adic L-function of $\tilde{\pi}$ (§6.6)
$\mathcal{L}_p^{\mathscr{C}}$ .. p-adic L-function in family over $\mathscr{C}$ (Def. 8.20)
Λ ... Localisation of $\mathcal{O}_\Omega$ at λ_π (§8.3.3)
λ_π (Pure, dominant, integral) weight of π (Def. 3.4)
$\mathfrak{m}_{\tilde{\pi}}$ ... Max. ideal in $\mathcal{H}$ (Def. 2.9)
$\mathfrak{m}_{\tilde{\pi}}^S$ ... Max. ideal in $\mathcal{H}^S$ (Def. 8.8)
$\mathfrak{m}_\lambda$ ... Max. ideal in $\mathcal{O}_\Omega$
Non-Q-critical ... Def. 3.14
Non-Q-critical slope ... Def. 3.15
N_Q ... Unipotent radical of Q
$v_{\lambda,j}$ ... Element of $V_\lambda(L)$ (§5.2)
Ω ... Affinoid in $\mathscr{W}_{\lambda_\pi}^Q$
Ω_π^ϵ ... Complex period (§2.10)
$\mathcal{O}_\Omega$ ... Ring of rigid functions on Ω
$\mathcal{O}_{F,p}$ ... $\mathcal{O}_F \otimes \mathbf{Z}_p$
p^β ... $\prod \mathfrak{p}^{\beta_\mathfrak{p}}$ (Def. 4.2)
pr_β ... Map $\pi_0(X_\beta) \to \mathscr{C}\ell_F^+(p^\beta)$ (5.3)
π ... Auto. repn. of $G(\mathbf{A})$ (Conditions 2.8)
$\tilde{\pi}$ ... p-refinement of π (§2.7, Conditions 2.8)
$\pi_0(X_\beta)$ ... Component group of auto. cycle (before (5.3))
ϖ_v ... Uniformiser of F_v
ψ ... Additive character of $F\backslash\mathbf{A}_F$ (§2.6)
Q ... Parabolic subgroup with Levi H
Q-refined RACAR .. Choice of Q-ref't $\tilde{\pi}_\mathfrak{p}$ $\forall \mathfrak{p}|p$ (§2.7)
$q_\mathfrak{p}$ ... $\mathrm{N}_{F/\mathbf{Q}}(\mathfrak{p})$
RACAR ... regular algebraic cuspidal auto. repn.
RASCAR ... η-symplectic RACAR (Intro.)
r_λ ... Specialisation map $\mathcal{D}_\lambda \to V_\lambda^\vee$ (eqn. (3.10))
Shalika model ... §2.6
Strongly non-Q-critical ... Def. 3.14
S_K .. Loc. symm. space for G of level K (eqn. (2.3))
S ... $\{v \nmid p\infty : \pi_v \text{ not spherical}\}$ (§2.4)
$\mathcal{S}_\psi^\eta$ ... Shalika model (§2.6)
sp_λ ... Any map induced from $\pmod{\mathfrak{m}_\lambda} : \mathcal{O}_\Omega \to L$
Σ ... Set of real embeddings of F
σ ... Real embedding $F \hookrightarrow \overline{\mathbf{Q}}$
$\sigma(\mathfrak{p})$ ... Real embedding attached to $\mathfrak{p}|p$ (§2.1)
$\mathbf{T}_{\Omega,h}$ ... Hecke algebra using $\mathcal{H}$ (Def. 7.1)
$\mathbf{T}_{\Omega,h}^S$ ... Hecke algebra using $\mathcal{H}^S$ (Def. 8.10)
$t = d(n^2+n-1)$ .. Top degree of cusp. cohomology
$t_\mathfrak{p}$ ... $\mathrm{diag}(\varpi_\mathfrak{p} I_n, I_n) \in \mathrm{GL}_{2n}(F_\mathfrak{p})$
t_p^β ... $\prod t_\mathfrak{p}^{\beta_\mathfrak{p}}$ (Def. 4.2)
$\tau(\chi_f)$ ... Gauss sum of χ_f (Thm. 5.22)
τ_β° ... Twisting map (§4.2.1)
$\Theta^{K,\epsilon}, \Theta_{i_p}^{K,\epsilon}$ ... Maps $\mathcal{S}_{\psi_f}^{\eta_f}(\pi_f^K) \to \mathrm{H}_\mathrm{c}^t$ (§2.10)
$U_\mathfrak{p} = U_\mathfrak{p}^\cdot$ ... Automorphic $U_\mathfrak{p}$-operator (§2.7)
$U_\mathfrak{p}^\circ$ ... Integrally normalised $U_\mathfrak{p}$ (§2.7,§3.3)
$\mathscr{U}(I)$ ... Integral ideles $\equiv 1 \pmod I$ (§2.1)
V_λ ... Alg. repn. of G of weight λ (§2.2)
V_λ^H ... Alg. repn. of H of weight λ
V_Ω^H ... (§3.2.3)
$\mathcal{V}$ ... Archimedean local system on S_K (§2.3.1)
$\mathscr{V}$ ... Non-archimedean local system on S_K (§2.3.2)
v_λ^H, v_Ω^H ... Elements of V_λ^H and V_Ω^H (Not. 5.9,6.3)
$\mathscr{W}_0$ ... Pure weight space (§3.1)
$\mathscr{W}_{\lambda_\pi}^Q$ ... (Parabolic) Weight space (Def. 3.4)
W^{FJ} ... Friedberg–Jacquet test vector (§2.6)
w (or $\mathsf{w}_\lambda, \mathsf{w}_\Omega$) ... Purity weight (§2.2)
w_n ... Longest Weyl element for GL_n (Def. 4.2)
X_β ... Automorphic cycle of level β (§4.1)
χ ... Finite order Hecke character
$\xi, \xi_\mathfrak{p}$ ... Twisting operator (Def. 4.2)
$x_{\tilde{\pi}}$ ... Point of $\mathscr{E}_{\Omega,h}$ corr. to $\tilde{\pi}$ (Thm. 7.6)
$x_{\tilde{\pi}}^S$ ... Point of $\mathscr{E}_{\Omega,h}^S$ corr. to $\tilde{\pi}$ (Thm. 8.11)
χ_{cyc} ... Cyclotomic character of Gal_p
Z ... Centre of G
$\mathbf{x}$ ... Element of $\mathscr{C}\ell_F^+(p^\beta)$
$*$-action of Δ_p ... §3.3
$\langle - \rangle_\lambda, \langle - \rangle_\Omega$ ... actions of $H(\mathbf{Z}_p)$ ((3.1),(3.5))
$-_{\mathfrak{m}_{\tilde{\pi}}}$ ... Localisation at $\mathfrak{m}_{\tilde{\pi}}$ (Def. 2.9)
$-_{\mathfrak{m}_{\tilde{\pi}}^S}$ ... Localisation at $\mathfrak{m}_{\tilde{\pi}}^S$ (Def. 8.8)

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