p-Adic Modular Forms

[caveat: huge topic, hard to know what to leave out]

§1: Classical modular forms

$\Gamma = \Gamma_1(N) \le SL_2(\mathbb{Z})$, k positive integer. equivalent ways of defining $M_k(\Gamma)$:

(Analytic) $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. M.f. is

$$f : \mathcal{H}^* \longrightarrow \mathbb{C} \quad \text{holomorphic,}$$

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \qquad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

($\mathbb{C}$-geometric) Let $pr : \mathcal{H}^* \longrightarrow \Gamma \backslash \mathcal{H}^* =: X_\Gamma$. Define sheaf ω_k on X_Γ by

$$\omega_k(V) := \{ f : pr^{-1}(V) \subset \mathcal{H}^* \to \mathbb{C} \text{ holo.} \mid f(\gamma\tau) = (c\tau+d)^k f(\tau) \}$$

$$\rightsquigarrow M_k(\Gamma) = \omega_k(X_\Gamma) = H^0(X_\Gamma, \omega_k).$$

Facts: - ω_k is a line bundle (loc. rk 1)
- both X_Γ and ω_k admit models over nice rings R

(Algo-Geometric) $M_k(\Gamma, R) = H^0(X_{\Gamma/R}, \omega_k)$

$\hookrightarrow$ have q-expansions in $R[[q]]$.

At heart of these results: moduli interpretation,

$$X_\Gamma(R) \sim \{ E/R \text{ ell. curve} + \text{"}\Gamma\text{-level structure"} \}.$$

(Algebraic) reinterpret: mod. form of wt k, lvl Γ over R is:

$$(1) \qquad f : \{ E/R + (\text{extra data}) \} \longrightarrow R$$

$$+ \text{ "functorial of wt } k\text{" in (extra data).}$$

§2: p-adic modular forms à la Serre

Guiding question: p prime, $f \in M_k(\Gamma_1(N), \overline{\mathbb{Q}})$. Let $k_m := k + p^m \to k$ p-adically.

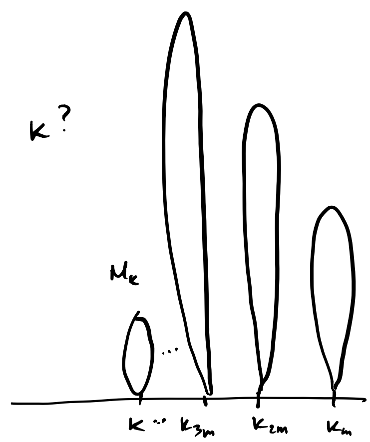
Q: Do $\exists$ Hecke eigenforms $f_m \in M_{k_m}(\Gamma_1(N), \overline{\mathbb{Q}})$ s.t. $f_m \to f$ as $m \to \infty$?

ie. $f(\tau) = \sum a_n q^n$, $f_m(\tau) = \sum a_n^m q^n$, and $a_n^m \to a_n$ $\forall n$? (uniformly)

Naive reinterpretation: (systematic approach)

Q: Can I p-adically deform $M_k(\Gamma)$ as I deform k?

A: No, for dimension reasons:



$\longrightarrow$ need to work with larger (∞-dim.) spaces.

First definition (Serre):

$$M_k^{p\text{-adic}}(\Gamma(1), \mathbb{Z}_p) = \left\{ f(q) \in \mathbb{Z}_p[[q]] : \exists f_i \in M_{k_i}(\Gamma(1), \mathbb{Z}) \text{ s.t. } f_i \to f,\ k_i \to k \right\}$$

= "p-adic completion of mod forms"

Remarks: 1) already very useful for studying congruences between m.f. (Ramanujan)

2) ... but this space is too big!

eg. $\lambda \in p\mathbb{Z}_p$, $f \in M^{p\text{-adic}}$, $f_\lambda := (1 - V_p\lambda)^{-1}(1 - V_p U_p) f \in M^{p\text{-adic}}$

Then $U_p f_\lambda = \lambda f_\lambda$

$\rightsquigarrow$ $\exists$ pathological eigenforms; spectrum of U_p is continuous

$\longrightarrow$ no good spectral theory of Hecke operators!

(can't expect $M^{p\text{-adic}}$ to tell us anything about classical eigenforms).

§3: Overconvergent modular forms

Coleman: geometric fix. Let $X = X_{SL_2(\mathbb{Z})}$.

Fact: $\exists$ modular form A over $\mathbb{Z}_p$ s.t.:

(1) $A(E, \text{data}) \in \begin{cases} \mathbb{Z}_p^\times & : E \text{ ordinary at } p, \\ p\mathbb{Z}_p & : E \text{ supersingular at } p, \end{cases}$

(2) $A(q) \equiv 1 \pmod{p}$ in $\mathbb{Z}_p[\![q]\!]$.

(If $p \geq 5$, can take $A = E_{p-1}$ Eisenstein).

Lemma: A is invertible in $M^{p\text{-adic}}(\Gamma(1), \mathbb{Z}_p)$.

Pf: $A^{p^n}(q) \equiv 1 \pmod{p^n}$

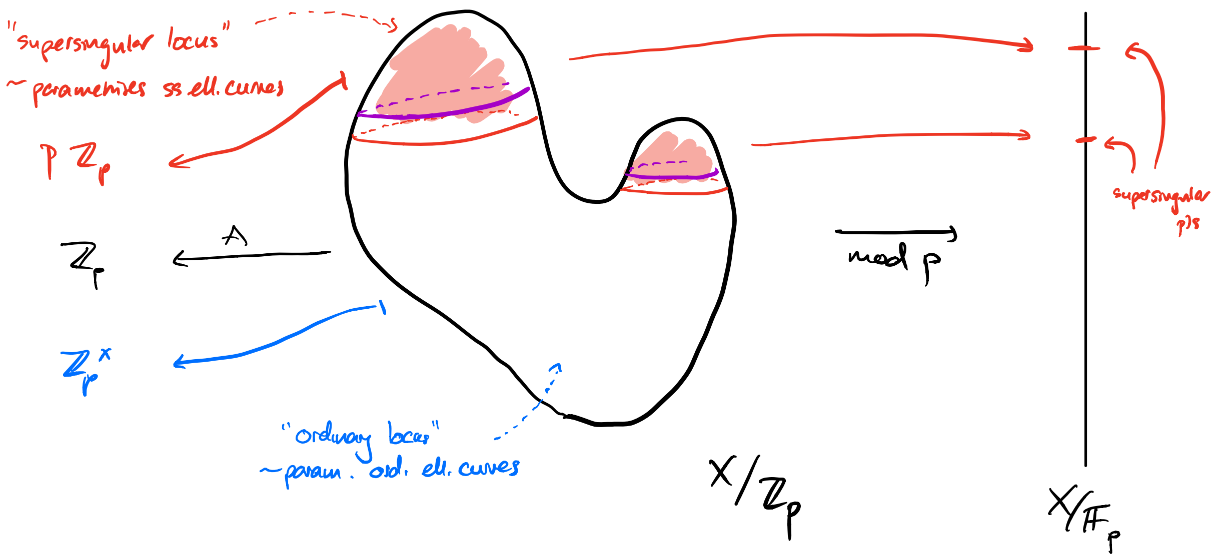
$\rightsquigarrow \lim_{n\to\infty} A^{p^n}(q) = 1$

$\rightsquigarrow \lim_{n\to\infty} A^{p^n-1}(q) = 1/A$. $\square$

Observe: E ss $\rightsquigarrow A(E, \text{data}) \in p\mathbb{Z}_p$ not invertible.

$\rightsquigarrow 1/A$ only well-defined on (E, data) with E ordinary.

Hence: want to make sense of "ordinary locus" $X^{ord} \subset X$.



$\rightsquigarrow$ X^{ord} = subspace where $|A| = 1$. (meaningless in Zariski... but definable in rigid world!)

Def'n: $(X/\mathbb{Z}_p, \omega_k) \overset{\text{GAGA}}{\rightsquigarrow} (\mathfrak{X}/\mathrm{Spf}\,\mathbb{Z}_p, \omega_k)$ formal scheme

Let $\mathfrak{X}^{ord} := \mathfrak{X}(|A|=1) \subseteq \mathfrak{X}$.

Thm: $M_k^{p\text{-adic}}(SL_2(\mathbb{Z}), \mathbb{Z}_p) \cong H^0(\mathfrak{X}^{ord}/\mathrm{Spf}\,\mathbb{Z}_p, \omega_k)$.

Have:

$$\begin{array}{ccc} M_k & = & H^0(\mathfrak{X}, \omega_k) \\ \cap & & \cap \\ M_k^{p\text{-adic}} & = & H^0(\mathfrak{X}^{ord}, \omega_k) \end{array}$$

(too small),

(too big).

Def'n: (Coleman). consider $0 \leq \varepsilon < \frac{p}{p+1}$

$$\mathfrak{X}^{ord} \subset \mathfrak{X}[\varepsilon] \subset \mathfrak{X}$$
$$\mathfrak{X}[\varepsilon] = \mathfrak{X}(|A| \geq |p^{\varepsilon}|).$$

~ parametrises ell. curves that are ordinary or "not too supersingular".

Define ε-overconvergent modular forms of wt k to be

$$M_k \subset M_k^{\dagger} := H^0(\mathfrak{X}[\varepsilon], \omega_k) \subset M_k^{p\text{-adic}}.$$

Facts: 1) $M_k^{\dagger}$ has discrete spectrum of Hecke eigenvalues

$\rightarrow$ not too big now!

2) after adding level $\Gamma_0(p)$-structure, spaces $M_k^{\dagger}$ vary p-adically in k.

$\longrightarrow$ do get eigenforms $f_m \to f$, for any f!

§4: Analytic o.c. mod. forms

	Analytic	Geometric
Classical	$f: \mathcal{H} \longrightarrow \mathbb{C}$	$H^0(X, \omega_k)$
Overconvergent		$H^0(\mathcal{X}[\varepsilon], \omega_k)$
p-adic (Serre)		$H^0(\mathcal{X}[0], \omega_k)$

Qs: 1) missing analytic def'ns?

2) more general p-adic weights?

$\mathbb{C}$-picture:

co-ord $z := \pi^* \xi$ on $\longleftarrow$ co-ord. ξ

func. $(cz+d)$ on

(defined using)

$$\begin{tikzcd} \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) \arrow[r, hook, "\pi"] \arrow[d, "pr"', "/\Gamma"] & \mathbb{P}^1(\mathbb{C}) \\ X_\Gamma & \end{tikzcd}$$

$\omega_k \longrightarrow X_\Gamma$

p-adic:

$z := \pi_{HT}^* \xi$ on $\longleftarrow$ co-ord ξ on

func $(cz+d)$ on

$$\begin{tikzcd} \mathcal{X}_{\Gamma(p^\infty)}[\varepsilon]_a \arrow[r, "\pi_{HT}"] \arrow[d, "pr"', "\text{“}/SL_2(\mathbb{Z}_p)\text{”}"] & \mathbb{P}^1 \\ \mathcal{X}[\varepsilon] & \end{tikzcd}$$

$\omega_k \longrightarrow \mathcal{X}[\varepsilon]$

Thm: (CHJ) An ε-overconvergent modular form is also a perfectoid function

$$f: \mathcal{X}_{\Gamma(p^\infty)}[\varepsilon]_a \longrightarrow \mathbb{C}_p$$

s.t.

$$f(\gamma\tau) = (cz+d)^{-k} f(\tau) \qquad \forall \gamma \in SL_2(\mathbb{Z}_p).$$

Def'n: a p-adic weight is a character

$$\kappa : \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times .$$

e.g. $k \in \mathbb{Z}$, $\quad k(x) = x^k$

$\rightsquigarrow$ k is a p-adic weight.

Def'n: κ a p-adic wt. Define

$$M_\kappa^\dagger(\Gamma_0(p)) := \Big\{ f : \mathcal{X}_{\Gamma(p^\infty)}[\varepsilon]_a \longrightarrow \mathbb{C}_p :$$

$$f(\gamma\tau) = \kappa^{-1}(cz+d) f(\tau)$$

$$\forall \gamma \in \Gamma_0(p) \Big\}.$$

Bonus: Hecke Operators

Have Hecke operators $\mathcal{U}_p$ and T_ℓ, $\ell \nmid p$, on mod. forms,

$$\text{e.g.} \quad \mathcal{U}_p f(q) = \sum a_{pn} q^n.$$

+ incredibly rich spectral theory: eigenforms.

But on p-adic m.f., too many eigenforms:

e.g. Let
$$V_p f(q) = \sum a_n q^{pn},$$
and f a p-adic m.f. Then $\mathcal{U}_p V_p = 1$. Let $\lambda \in p\mathbb{Z}_p$, and

$$\begin{aligned} f_\lambda &= (1-\lambda V_p)^{-1}(1-V_p\mathcal{U}_p)f \\ &= \Big(\sum_{n\geq 0} \lambda^n V_p^n\Big)(1-V_p\mathcal{U}_p)f, \end{aligned}$$
exists as a p-adic m.f.

Then
$$\begin{aligned} \mathcal{U}_p f_\lambda &= \Big(\sum_{n\geq 0} \lambda^{n+1} V_p^n\Big)(1-V_p\mathcal{U}_p f) + (\mathcal{U}_p - \underbrace{\cancel{\mathcal{U}_p V_p}}_{1}\mathcal{U}_p)f \\ &= \lambda f_\lambda \qquad + \qquad 0 \\ &= \lambda f_\lambda. \end{aligned}$$