# Describing the Universal Deformation Ring

*These are notes for a talk I gave at the Warwick Number Theory study group on Deformations of Galois representations, February 2015, and were written mainly for my own benefit. The results stated make more sense when taken in step with the rest of the study group. These notes are largely taken from [Gou01].*

## 1. Introduction and Recap

Recall our work so far; we have developed the basic theory of Galois deformations and, in the previous lecture, have shown (modulo Schlessinger's generalisation of a theorem of Grothendieck) that the deformation functor is representable. More precisely, let $\Pi$ be a profinite group, $k$ a finite field of characteristic $p$ and

$$\overline{\rho} : \Pi \longrightarrow \mathrm{GL}_n(k)$$

an absolutely irreducible residual representation. (In fact, absolute irreducibility is a stronger condition than we actually need; it suffices to say that the only $\Pi$-module endomorphisms of $k^n$ are scalars. From now on, we'll assume that *all* of our residual representations have this property).

**Definition 1.1.** Let $\mathcal{C}$ be the category of complete local noetherian rings with residue field $k$, with morphisms being local homomorphisms that induce the identity on residue fields. We call the objects *coefficient rings*, and they are quotients of power series over $W(k)$, the ring of Witt vectors for $k$. For an object $\Lambda$ of $\mathcal{C}$, define the category $\mathcal{C}_\Lambda$ to be the category whose objects are complete local noetherian $\Lambda$-algebras with residue field $k$, with the same morphisms. Then we have $\mathcal{C} = \mathcal{C}_{W(k)}$.

For $A$ an object of $\mathcal{C}$, a *deformation of* $\overline{\rho}$ *to* $A$ is a strict equivalence class of homomorphisms $\rho : \Pi \to \mathrm{GL}_n(A)$ such that the reduction of $\rho$ to $k$ gives $\overline{\rho}$. We define a functor $D_\Lambda : \mathcal{C}_\Lambda \to \mathit{Set}$ by setting $D_\Lambda(A) := \{\text{deformations of } \overline{\rho} \text{ to } A\}$. To this, we associate the *Zariski tangent space* $t_{D_\Lambda} = D_\Lambda(k[\varepsilon])$, a finite dimensional $k$-vector space, where $k[\varepsilon] := k[X]/[X^2]$.

**Theorem 1.2** (Mazur, Ramakrishna)**.** *The functor $D_\Lambda$ is representable. In particular, there exists a* universal deformation ring $R^{\mathrm{univ}} = R^{\mathrm{univ}}(\overline{\rho}) \in \mathrm{Obj}(\mathcal{C}_\Lambda)$ *and a* universal deformation

$$\rho^{\mathrm{univ}} : \Pi \longrightarrow \mathrm{GL}_n(R^{\mathrm{univ}})$$

*such that for any object $A \in \mathcal{C}_\Lambda$ and deformation $\rho : \Pi \to \mathrm{GL}_n(A)$, there is a map $R^{\mathrm{univ}} \to A$ which sends $\rho^{\mathrm{univ}}$ to $\rho$. Moreover, if $d := \dim_k t_{D_\Lambda}$, we can describe $R^{\mathrm{univ}}$ as*

$$R^{\mathrm{univ}} = \Lambda[[X_1, ..., X_d]]/I$$

*for some ideal $I$.*

In this talk, we aim to describe the universal deformation ring in more detail. Firstly, we compute this ring in the case where $\overline{\rho}$ is a character, that is, for $n = 1$; secondly, we look at the number of generators $d$, showing that we can describe this in cohomological terms.

# 2. The Universal Character

In this section, we focus on the case where $n = 1$, where we can obtain a complete description of the theory. With this in mind, let $\overline{\chi} : \Pi \to k^\times$ be our residual representation. We are looking to characterise all lifts to $A^\times$ for objects $A$ of $\mathcal{C}_\Lambda$. First note that via the Teichmuller lift, we have a canonical decomposition

$$A^\times \cong k^\times \times (1 + \mathfrak{m}_A),$$

where $\mathfrak{m}_A$ is the maximal ideal of $A$. In particular, we have a canonical lift of $\overline{\chi}$ to $\chi_A : \Pi \to A^\times$, the *Teichmuller lift.*

Now let $\chi : \Pi \to A^\times$ be an arbitrary lift of $\overline{\chi}$ to $A$. As $\chi$ lifts $\overline{\chi}$, the character $\chi$ is entirely determined by its values projected down to $1 + \mathfrak{m}_A$, that is, we can write $\chi = (\overline{\chi}, \psi)$, where $\psi$ is a homomorphism $\Pi \to 1 + \mathfrak{m}_A$. Now, $1 + \mathfrak{m}_A$ is an abelian pro-$p$ group (that is, every open normal subgroup has $p$-power index); indeed,

$$1 + \mathfrak{m}_A \cong \varprojlim (1 + \mathfrak{m}_A)/(1 + \mathfrak{m}_A^n),$$

and each term in the inverse limit is isomorphic to a finite abelian $p$-group. Any homomorphism from a group to an abelian pro-$p$ group must factor through the pro-$p$ completion of its abelianisation. So $\psi$ factors through this quotient $\Gamma$ of $\Pi$. In other words, we have a canonical map $\Pi \to \Gamma$ and a map $\widetilde{\psi} : \Gamma \to 1 + \mathfrak{m}_A$ associated to $\psi$ such that any $\psi$ factors as

$$\psi : \Pi \longrightarrow \Gamma \longrightarrow 1 + \mathfrak{m}_A. \tag{1}$$

**Examples:** (i) Suppose $\Pi = G_{\mathbb{Q},p}$. Then from class field theory, we know that $\Pi^{\mathrm{ab}} \cong \mathbb{Z}_p^\times$, and the pro-$p$ completion of of $\mathbb{Z}_p^\times$ is $1 + p\mathbb{Z}_p$. Thus in this case $\Gamma = 1 + p\mathbb{Z}_p$.

(ii) If $\Pi = G_{\mathbb{Q}_\ell}$ for some prime $\ell$, then we have $\Pi^{\mathrm{ab}}$ is $\widehat{\mathbb{Z}}$, and thus $\Gamma = \mathbb{Z}_p$.

Equation (1) is akin to the universal property we are looking for. Accordingly, it makes sense to try and construct a coefficient ring out of $\Gamma$. There is a natural way of doing this; namely, we take the completed group ring

$$\Lambda[[\Gamma]] := \varprojlim \Lambda[\Gamma/U],$$

where the limit runs over all open normal subgroups of $\Gamma$.

**Remark:** The notation here is slightly confusing; at first glance, it looks like this ring could be an 'infinite sum' analogue of the usual group rings. If this were so, we'd expect the element $\sum_{g\in\Gamma}[g]$ to live in $\Lambda[[\Gamma]]$, but this does not happen. Indeed, there is a natural *augmentation map* $\Lambda[[\Gamma]] \to \Lambda[\Gamma/\Gamma] \cong \Lambda$, given by mapping every basis element $[g]$ to 1. Clearly the 'element' given earlier does not have well-defined image under the augmentation map. So we must impose *at least* some conditions on the coefficients of our power series.

**Proposition 2.1.** *The ring* $\Lambda[[\Gamma]]$ *is an object of* $\mathcal{C}_\Lambda$*.*

*Proof.* If $I$ denotes the kernel of the augmentation map, then the maximal ideal is $(I, \mathfrak{m}_\Lambda)$. For a complete proof, see [Sha]. □

**Theorem 2.2.** *For a residual character, we have $R^{\text{univ}} = \Lambda[[\Gamma]]$. The universal character is given by*

$$\chi^{\text{univ}} : \Pi \longrightarrow \Lambda[[\Gamma]]^\times,$$
$$g \longmapsto \chi_\Lambda(g)[\gamma(g)],$$

*where $\gamma : \Pi \to \Gamma$ is the projection to $\Gamma$.*

*Proof.* It's clear that $\chi^{\text{univ}}$ is a character. It remains to show that it is universal. To this end, let $\chi : \Pi \to A^\times$ be a lift of $\overline{\chi}$ to a coefficient ring $A$. Then, from above, we know that $\chi$ gives rise to a map $\widetilde{\psi} : \Gamma \to 1 + \mathfrak{m}_A$. This, together with the $\Lambda$-algebra structure on $A$, induces a map

$$\Lambda[[\Gamma]] \longrightarrow A.$$

(Indeed, there is an obvious map $\Lambda[\Gamma] \to A$; complete this map with respect to the corresponding maximal ideals). We then see easily that $\widetilde{\psi} \circ \chi^{\text{univ}} = \chi$, and we are done. □

Note here that the universal deformation ring in the 1-dimensional case is thus *independent* of the initial character. This is part of a wider phenomenon; indeed, in [Maz89], it is proved that:

**Theorem 2.3.** *Let $\overline{\rho}$ and $\overline{\rho}'$ be residual representations with universal deformation rings $R^{\text{univ}}(\rho)$ and $R^{\text{univ}}(\rho')$, and suppose that $\overline{\rho} \cong \overline{\rho}' \otimes \overline{\chi}$ for a residual character $\overline{\chi}$. Then there is a canonical isomorphism*

$$\phi : R^{\text{univ}}(\rho) \cong R^{\text{univ}}(\rho'),$$

*and $\phi$ takes $\rho^{\text{univ}}$ to $(\rho')^{\text{univ}} \otimes \chi_0$, where $\chi_0$ is the Teichmuller lift of $\overline{\chi}$ to $\Lambda$.*

We end this section with a remark on wider applications of the case of characters. Indeed, suppose $\overline{\rho}$ is a residual representation to $\mathrm{GL}_n(k)$, with universal deformation ring $R^{\text{univ}}(\rho)$; then we can apply the determinant map to get a character $\overline{\chi} = \det(\overline{\rho})$. Any deformation $\rho$ of $\overline{\rho}$ will give a deformation $\det(\rho)$ of $\overline{\chi}$, and then naturally we see that applying this to the universal deformation of $\overline{\rho}$, we get a deformation

$$\det(\rho^{\text{univ}}) : \Pi \to (R^{\text{univ}})^\times$$

of $\overline{\chi}$. Thus there is a natural map from the universal deformation ring $\Lambda[[\Gamma]]$ of $\overline{\chi}$ to $R^{\text{univ}}$, giving the latter a $\Lambda[[\Gamma]]$-algebra structure.

# 3. The Relation to Cohomology

We can give more complete descriptions of the universal deformation ring via the group cohomology of $\Pi$. Indeed, recall that we can view $R^{\text{univ}}$ as a quotient of a power series ring in $d$ variables over $\Lambda$, where $d = \dim_k t_{D_\Lambda} = \dim_k D_\Lambda(k[\varepsilon])$. We will show that we have an isomorphism $t_{D_\Lambda} = \mathrm{H}^1(\Pi, \mathrm{Ad}(\overline{\rho}))$, where $\mathrm{Ad}(\overline{\rho})$ is the adjoint representation of $\rho$, and that this space is *further* naturally isomorphic to the group $\mathrm{Ext}_\Pi(\overline{\rho}, \overline{\rho})$ of extensions of $\overline{\rho}$ by itself.

## 3.1. The Relation to $\mathrm{Ad}(\overline{\rho})$

Suppose $\overline{\rho}$ is a residual representation with universal deformation ring $R^{\mathrm{univ}}$. Consider an element $\rho \in D_\Lambda(k[\varepsilon])$. Take a representative homomorphism $\phi : R^{\mathrm{univ}} \to k[\varepsilon]$. Then as $\phi$ lifts $\overline{\rho}$, we know we can write

$$\phi(g) = (1 + b_g\varepsilon)\overline{\rho}(g),$$

where $b_g \in \mathrm{M}_n(k)$. Then

$$\begin{aligned}\phi(gh) = \phi(g)\phi(h) &= (1 + b_g\varepsilon)\overline{\rho}(g)(1 + b_h\varepsilon)\overline{\rho}(h)\\ &= (1 + (b_g + \overline{\rho}(g)b_h\overline{\rho}(g)^{-1})\varepsilon)\overline{\rho}(gh).\end{aligned}$$

It follows that if we let $\Pi$ act on $\mathrm{M}_n(k)$ by

$$g \cdot A = \overline{\rho}(g)A\overline{\rho}(g)^{-1},$$

that we have

$$b_{gh} = b_g + g \cdot b_h,$$

that is, the association $g \mapsto b_g$ is a cocycle. We call $\mathrm{M}_n(k)$ with this action of $\Pi$ the *adjoint representation of* $\overline{\rho}$, and denote it $\mathrm{Ad}(\overline{\rho})$.

**Proposition 3.1.** *The association $g \mapsto b_g$ induces an isomorphism*

$$D_\Lambda(k[\varepsilon]) \cong \mathrm{H}^1(\Pi, \mathrm{Ad}(\overline{\rho}))$$

*of $k$-vector spaces.*

*Proof.* This is a simple exercise. One must show that strictly equivalent homomorphisms give cocycles that differ by a coboundary; for this, one uses the relation that if $N \in \mathrm{M}_n(k)$, then $(1 + N\varepsilon)^{-1} = (1 - N\varepsilon) \in \mathrm{GL}_n(k[\varepsilon])$, and then it follows that the cocycles given by $\phi$ and $(1+N\varepsilon)\phi(1-N\varepsilon)^{-1}$ differ by the coboundary $N - g\cdot N$. As this process can be reversed, we have injectivity. Likewise, from a cocycle we can easily construct a homomorphism lifting $\overline{\rho}$, proving surjectivity. That this is a $k$-vector space map is easily shown from the definitions and is more instructive to work out yourself. ☐

**Corollary 3.2.** *There is a short exact sequence*

$$0 \longrightarrow I \longrightarrow \Lambda[[X_1, ..., X_d]] \longrightarrow R^{\mathrm{univ}} \longrightarrow 0,$$

*where $d = \dim_k \mathrm{H}^1(\Pi, \mathrm{Ad}(\overline{\rho}))$.*

Why is this important? When $\Pi$ is a Galois group, we are brought into the realms of Galois cohomology, bringing a myriad of tools to our disposal. For example, in the next talk, we'll show that the ideal $I$ that we quotient by to obtain $R^{\mathrm{univ}}$ can be generated by at most $d_2 = \dim_k \mathrm{H}^2(\Pi, \mathrm{Ad}(\overline{\rho}))$ elements, so that we have neat lower bounds for the dimension of $R^{\mathrm{univ}}$ given by the Euler-Poincaré characteristic.

## 3.2. The Relation to $\mathrm{Ext}_\Pi(\overline{\rho}, \overline{\rho})$

We have a further description. The study of Galois deformations is, in a sense, looking to find families of Galois deformations. One of the simplest ways of doing this is to talk about *extensions* of representations. The tangent space also characterises certain types of these extensions; we make this more precise in the following.

Let $\overline{\rho}$ be a residual respresentation and consider $\rho \in D_\Lambda(k[\varepsilon])$, represented by a homomorphism $\phi$. Denote by $V_{\overline{\rho}}$ the space $k^n$ with the action of $\Pi$ given by $\overline{\rho}$, and by $E$ the space $k[\varepsilon]^n$ with the action of $\Pi$ given by $\phi$. Then we have a short exact sequence

$$0 \longrightarrow V_{\overline{\rho}} \xrightarrow{\alpha} E \xrightarrow{\beta} V_{\overline{\rho}} \longrightarrow 0,$$

where we identify the first copy of $V_{\overline{\rho}}$ with $\varepsilon E$ and let $\alpha$ be the natural inclusion, and the second copy with the quotient $E/\varepsilon E$. Thus any element of $D_\Lambda(k[\varepsilon])$ determines an extension of $\overline{\rho}$ by itself.

Conversely, suppose we are given an extension

$$0 \longrightarrow V_{\overline{\rho}} \xrightarrow{\alpha} E \xrightarrow{\beta} V_{\overline{\rho}} \longrightarrow 0$$

of $\Pi$-modules. Then we want to find a $k[\varepsilon]$-module structure on $E$. Indeed, define multiplication by $\varepsilon$ by the map $\alpha \circ \beta$; we see from exactness that $(\alpha \circ \beta)^2 = 0$, hence this is well-defined. We also see that as $\alpha$ and $\beta$ are $\Pi$-module homomorphisms, this multiplication commutes with the action of $\Pi$. It turns out that this construction turns $E$ into a free $k[\varepsilon]$-module with an action of $\Pi$, thus defining a representation $\Pi \to \mathrm{GL}_n(k[\varepsilon])$. A little more work shows:

**Proposition 3.3.** *There are canonical isomorphisms*

$$t_{D_\Lambda} := D_\Lambda(k[\varepsilon]) \cong \mathrm{H}^1(\Pi, \mathrm{Ad}(\overline{\rho})) \cong \mathrm{Ext}_\Pi(\overline{\rho}, \overline{\rho})$$

*of $k$-vector spaces.*

# References

[Gou01] Fernando Gouvêa. Deformations of Galois representations. In *Arithmetic Algebraic Geometry*. AMS and IAS/Park City Mathematics Institute, 2001.

[Maz89] Barry Mazur. Deforming Galois representations, 1989.

[Sha] Romyar Shafiri. Notes on Iwasawa theory. Lecture notes.