P-adic Asai L-functions of Bianchi modular forms

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These are notes from a talk I gave at a Barcelona Number Theory seminar in May 2017. The results described here are a report on joint work with David Loeffler, in which we construct a p-adic Asai L-function attached to a weight 2 ordinary Bianchi modular form using techniques from the theory of Euler systems. The results will appear in a forthcoming paper titled 'P-adic Asai L-functions for Bianchi modular forms'. *Disclaimer: I have been loose with the details, and ignored certain technical points to ease exposition.*

Motivation

I'll start with the following uncontroversial mantra:

"L-functions carry important arithmetic information."

If one is prepared to believe this, then the logical conclusion one is forced to draw is;

"If you can find a well-behaved L-function, then it is worth studying."

In other words, given such an L-function, what information can you mine from it? And what tools does one need to get at it?

An increasingly productive tool for linking arithmetic data and L-functions is the use of p-adic methods, and in particular, the construction and study of p-adic L-functions. (Of this, more later). So a natural follow-up conclusion is:

"Given a well-behaved L-function, it is worth constructing a p-adic version of it."

Example: *(Classical modular forms).* We start with the following well-known example. Let $\Gamma = \Gamma_1(N) \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, and define $Y_1^{\mathbb{Q}}(N) := \Gamma_1(N)\backslash\mathcal{H}$. A *modular form of weight 2 and level* Γ is (loosely) either:

(a) An automorphic form for $\mathrm{GL}_2/\mathbb{Q}$ of weight 2 and level Γ,

(b) A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ invariant under the weight 2 action of Γ, or

(c) A cohomology class $\phi_f \in \mathrm{H}^1_{\mathrm{c}}(Y_1^{\mathbb{Q}}(N), \mathbb{C})$.

On such objects, we have a Hecke action; suppose that $T_n f = a_n f$ for all n. Then for a Dirichlet character χ we can define the L-function of f to be

$$L(f, \chi, s) := \sum_{n\geq 0} a_n \chi(n) n^{-s}.$$

Conjecturally, the special values of this L-function are linked to the arithmetic of f; for example, if f corresponds to an elliptic curve E, then the Birch and Swinnerton-Dyer conjecture links $L(f, 1)$ to the rank of E (and, via the leading term, to the order of its torsion subgroup and Tate-Shafarevich group, its Tamagawa numbers, and its regulator).

Any conjecture that lists BSD as a special case must be extremely difficult, and the family of conjectures (e.g. Bloch-Kato, Beilinson) that link L-functions to arithmetic are no different. Let's suppose, however, we want to prove the rank part of BSD. Then we can split this into two natural pieces:

- Step 1: Prove that $\mathrm{rank}(E/\mathbb{Q}) \leq \mathrm{ord}_{s=1}L(f,s)$. Then we want to bound the rank. One method for doing this is to bound Selmer groups, and one method for bounding Selmer groups is to use the powerful theory of Euler systems. It turns out that one of the best ways of proving that Euler systems are non-trivial – and hence useful – is to use p-adic L-functions.
- Step 2: Prove that $\mathrm{ord}_{s=1}L(f,s) \leq \mathrm{rank}(E/\mathbb{Q})$. A sensible approach here to try and construct points on the curve. This has been done using archimedean methods (for example, Heegner points) and, more recently, using p-adic methods in the work of Henri Darmon, Victor Rotger and their many collaborators. These constructions make heavy use of p-adic L-functions.

So, in both directions, p-adic L-functions have been at the forefront of recent advances. Given their importance, it's desirable to construct them in as wide a generality as possible. In this setting:

Theorem (Many, many people)**.** *The p-adic L-function of f exists. (See [MSD74], [MTT86], [PS11], and [Bel12] for various constructions).*

Example: *(Bianchi modular forms).* The above serves as a precursor to the theory over imaginary quadratic fields. Let F be such an imaginary quadratic field, and let $\mathcal{H}_3 := \mathbb{C}\times\mathbb{R}_{>0}$ denote the upper half-space. Now let $\Gamma = \Gamma_1(\mathfrak{n}) \subset \mathrm{SL}_2(\mathcal{O}_F)$ be a congruence subgroup, and define $Y_1^F(\mathfrak{n}) := \Gamma\backslash\mathcal{H}_3$. A *Bianchi modular form of weight 2 and level* Γ is either:

(a) An automorphic form for GL_2/F of weight 2 and level Γ,

(b) A harmonic function $f : \mathcal{H}_3 \to \mathbb{C}^3$ invariant under the weight 2 action of Γ, or

(c) A cohomology class $\phi_f \in \mathrm{H}^1_{\mathrm{c}}(Y_1^F(\mathfrak{n}), \mathbb{C})$.

Again, one has a Hecke action, now indexed by non-zero ideals in the ring of integers $\mathcal{O}_F$. Suppose again that $T_I f = a_I f$ for all such I. Then for a Hecke character χ of F, we define

$$L(f,\chi,s) := \sum_{0\neq I\subset\mathcal{O}_F} a_I\chi(I)N(I)^{-s}.$$

Again, conjecturally, special values of this L-function see important information (for example the Selmer group of the associated Galois representation). Also conjecturally Bianchi modular forms of weight 2 have strong links to elliptic curves over F, though this picture is murky and the conjectures are even harder to establish in this setting. We do, however, have:

Theorem. *For most f, the p-adic L-function of f exists (see [Wil17]).*

This was one of the main results of my PhD thesis. I have an ongoing joint project with Daniel Barrera which might, as a consequence, allow 'most' to be replaced with 'most plus some more' in the above. (Replacing 'most' with 'all' seems a long way away at this stage, however).

For the purposes of this talk, though, we are not interested in this L-function – the 'standard' L-function – but rather another L-function attached to f by Asai in 1977, the *Asai* or *twisted tensor* L-function. This is defined rather curiously by a brutal 'restriction' of the standard L-function to rational ideals, that is, to the integers. In particular, for a (classical) Dirichlet character χ define:

$$L^{\mathrm{As}}(f,\chi,s) := L(\psi_f, 2s-2)\sum_{n\geq 0} a_{(n)}\chi(n)n^{-s},$$

where ψ_f is the character of f. (Disclaimer: this definition requires a little more tweaking to be entirely accurate). This turns out to be a well-behaved L-function, and in the spirit of the above, it's worth studying and we should try and find a p-adic version!

The Asai L-function

The Asai L-function has (see [Asa77]):

(1) An Euler product,

(2) A functional equation, and

(3) Meromorphic continuation, with a possible pole only at $s = 1$.

We 'expect' an L-function to carry some arithmetic information in its special values, and there is such a link here: it has been proved that:

Fact: The Asai L-function of f has analytic continuation to $s = 1$ if and only if f is *not* a base-change from $\mathbb{Q}$. In other words, $L^{\mathrm{As}}(f, s)$ encodes information about base-change functoriality. (I don't know a good reference for this; it seems to be somewhat scattered around the literature. I'd be very happy to be informed of one).

The Asai L-function has a natural description in terms of Galois representations. Let

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow \mathrm{Aut}(V)$$

denote the (two dimensional) Galois representation attached to f, and consider the tensor product

$$\rho_f \otimes \rho_{\overline{f}} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow \mathrm{Aut}\big(V \otimes \overline{V}\big).$$

This extends to (four dimensional) representations $\rho^{\pm}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in two natural ways, and (up to normalisation) we have

$$L(\rho^+, s) = L^{\mathrm{As}}(f, s) \quad \text{and} \quad L(\rho^-, s) = L^{\mathrm{As}}(f, \chi_{F/\mathbb{Q}}, s),$$

where $\chi_{F/\mathbb{Q}}$ is the character of the quadratic extension $\mathbb{F}/\mathbb{Q}$. For more on this, see [Gha99], Section 4, for a treatment in (conjectural) motivic terms.

Our aim, in the spirit of slogan 2, is to construct a p-adic version of the Asai L-function.

Strategy

Constructions of p-adic L-functions from cohomology classes tend to come in similar flavours, and there are usually three main steps (each involving significant amounts of work). Loosely, these are:

(1) Explicitly link cohomology classes to L-values,

(2) Find an integral version,

(2) Interpolate this link by factorising through distributions.

More precisely, in our case one could hope for the following.

(1) For each Dirichlet character χ, construct a map

$$\mathrm{Ev}_\chi : \mathrm{H}^1_c(Y_1^F(\mathfrak{n}), \mathbb{C}) \longrightarrow \mathbb{C}$$

such that

$$\mathrm{Ev}_\chi(\phi_f) = (*) L^{\mathrm{As}}(f, \chi, 1),$$

for some explicit factor $(*)$ (that could be zero, and indeed is zero precisely when the Dirichlet character is odd).

(2) Replace $\mathbb{C}$ with the ring of integers $\mathcal{O}$ in some number field (or finite extension of $\mathbb{Q}_p$).

(3) Find a map
$$\mu : \mathrm{H}^1_{\mathrm{c}}(Y_1^F(\mathfrak{n}), \mathcal{O}) \longrightarrow \mathcal{O}[[\mathbb{Z}_p^\times]]$$
such that if χ has p-power conductor, Ev_χ factors as
$$\mathrm{H}^1_{\mathrm{c}}(Y_1^F(\mathfrak{n}), \mathcal{O}) \xrightarrow{\ \mu\ } \mathcal{O}[[\mathbb{Z}_p^\times]] \xrightarrow{\ [\chi]\ } \mathcal{O}(\chi) \subset \overline{\mathbb{Q}},$$
where $\mathcal{O}(\chi)$ is the extension of $\mathcal{O}$ obtained by adjoining the values of χ and $[\chi]$ is the map obtained by sending $[a] \to \chi(a)$ and extending linearly.

Given this, we end up with a canonical element $L_p^{\mathrm{As}} := \mu(\phi_f) \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \otimes_{\mathbb{Z}} \mathcal{O}$, which we view as a distribution on $\mathbb{Z}_p^\times$, and then if $\mathrm{cond}(\chi) = p^r$, we have
$$\begin{aligned} L_p^{\mathrm{As}}(\chi) &:= [\chi](L_p) = [\chi] \circ \mu(\phi_f) \\ &= \mathrm{Ev}_\chi(\phi_f) = (*) L^{\mathrm{As}}(f, \chi, 1). \end{aligned}$$

Thus we've defined a distribution on $\mathbb{Z}_p^\times$ that interpolates critical values of the Asai L-function - that is, a p-adic Asai L-function!

We follow this strategy. Part (1) can be summarised as (work of Ghate)+ε, in the sense that Ghate (in his PhD thesis) constructed the map Ev_1 for the trivial character, and David and I extended to non-trivial characters using his methods. Part (2) is similarly a combination of old and new. Working out part (3), however, is entirely new and forms the meat of our results.

Ghate's results: constructing Ev_χ

Ghate's work fits into a somewhat larger framework that can be summarised as:

L-values can often be obtained by pairing things with Eisenstein series.

For example, the Rankin-Selberg method refers to a technique where one obtains an integral formula for an L-function by pairing with an Eisenstein 'thing' as above. The L-function then often inherits additional structure from the Eisenstein object, such as the functional equation and analytic continuation. In short, we should look for a suitable Eisenstein object to pair our cohomology class with.

A second observation is that compactly supported cohomology is naturally dual to Betti cohomology in a complementary degree. In this case, $Y_{F,\Gamma}$ is a three-dimensional real manifold, so the complementary degree of $\mathrm{H}^1_{\mathrm{c}}$ is 2, and we have a pairing
$$\langle \cdot \rangle : \mathrm{H}^1_{\mathrm{c}}(Y_1^F(\mathfrak{n}), \mathbb{C}) \otimes \mathrm{H}^2(Y_1^F(\mathfrak{n}), \mathbb{C}) \longrightarrow \mathbb{C}$$
induced by cup product. Hence we're looking for 'Eisenstein things' in H^2.

A further observation is that from the natural inclusion $Y_1^{\mathbb{Q}}(N) \hookrightarrow Y_1^F(\mathfrak{n})$, where $(N) = \mathfrak{n} \cap \mathbb{Z}$, we have a pushforward
$$\iota : \mathrm{H}^1(Y_1^{\mathbb{Q}}(N), \mathbb{C}) \longrightarrow \mathrm{H}^2(Y_1^F(\mathfrak{n}), \mathbb{C}).$$
So if we can find an Eisenstein class in $\mathrm{H}^1(Y_1^{\mathbb{Q}}(N), \mathbb{C})$ then we can push it forward into the cohomology group we want. But these things are well understood! Indeed:

Definition. (Sketch). Define a differential on $\mathcal{H}$ by
$$E_N(\tau) := (*) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1(N)} \gamma_* \left(\frac{d\tau}{\mathrm{Im}(\tau)} \right),$$
where $(*)$ is some explicit factor and Γ infinity is the stabiliser of $\infty \in \mathbb{P}^1(\mathbb{Q})$ in $\Gamma_1(N)$.

This descends to a differential on $Y_1^{\mathbb{Q}}(N)$, and gives a class in $\mathrm{H}^1_{\mathrm{dR}}(Y_1^{\mathbb{Q}}(N), \mathbb{C})$. Under a comparison isomorphism we get a class in the Betti cohomology (which we will denote by the same thing), and then in [Gha99] Ghate proves[1]:

Theorem (Ghate)**.** *We have*

$$\langle \phi_f, \iota_* E_N \rangle = (*) L^{\mathrm{As}}(f, 1).$$

In particular, we can define

$$\mathrm{Ev}_1(\phi) = \langle \phi, \iota_* E_N \rangle.$$

What about non-trivial characters (the ε of above)? Using much the same idea, David and I extended this. In particular, let a be some class $\pmod{p^r}$, and define

$$\beta_{a,r} := \begin{pmatrix} 1 & a/p^r \\ 0 & 1 \end{pmatrix}.$$

This defines a map $Y_1^F(\mathfrak{n}p^{2r}) \to Y_1^F(\mathfrak{n})$, and we define

$$E_{N,r}^{a,F} := (\beta_{a,r})_* \iota_* E_{Np^{2r}}.$$

Then David and I showed that:

Proposition. *Suppose the conductor of χ is p^r (though this goes through for general conductor). Define*

$$\mathrm{Ev}_\chi(\phi) := \sum_{a \pmod{p^r}} \chi(a) \langle \phi, E_{N,r}^{a,F} \rangle.$$

Then

$$\mathrm{Ev}_\chi(\phi_f) = (*) L^{\mathrm{As}}(f, \chi, 1).$$

Interpolating the Ev_χ

The real meat of the new work comes in the second part of our strategy, namely p-adically interpolating the Ev_χ. For now, let's assume that the first part goes through, that is, we can find some finite extension $\mathcal{O}$ of $\mathbb{Z}_p$ and work integrally.

Suppose that χ has conductor p^r. Then observe that the map

$$[\chi] : \mathcal{O}[[\mathbb{Z}_p^\times]] \longrightarrow \mathcal{O}(\chi)$$

factors through the finite level, that is the group ring $\mathcal{O}[(\mathbb{Z}/p^r)^\times]$. Moreover we can easily factor Ev_χ through this as well, as

$$\mathrm{H}^1_{\mathrm{c}}(Y_1^F(\mathfrak{n}), \mathcal{O}) \xrightarrow{\ \mu_r\ } \mathcal{O}[(\mathbb{Z}/p^r)^\times] \xrightarrow{\ [\chi]\ } \mathcal{O}(\chi),$$

where

$$\mu_r(\phi) := \sum_{a \pmod{p^r}} \langle \phi, E_{N,r}^{a,F} \rangle [a].$$

Rearranging, we can see μ_r as the map

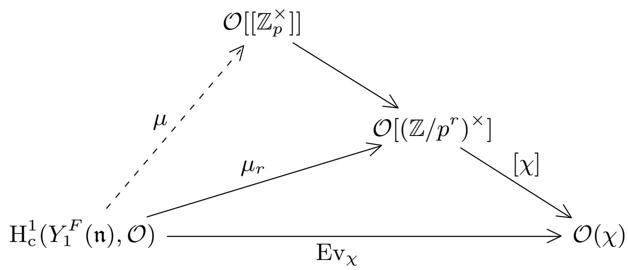
$$\mu_r(\phi) = \langle \phi, \Xi_{N,r} \rangle,$$

[1] Ghate phrases this slightly differently. In particular, he takes the pullback of the Bianchi class to $\mathcal{H}$ rather than the pushforward of the Eisenstein class to $\mathcal{H}_3$. But these calculations are really just two sides of the same coin.

where

$$\Xi_{N,r} := \sum_{a \pmod{p^r}} E_{N,r}^{a,F} \otimes [a] \quad \in \mathrm{H}^2(Y_1^F(\mathfrak{n}), \mathcal{O}) \otimes \mathcal{O}[(\mathbb{Z}/p^r)^\times].$$

We now have the following picture:



Here, the dashed line is the map we're aiming to construct, whilst the top right map is simply the natural quotient map. A natural thing to do now is to try and arrange the various μ_r into an inverse limit over r to obtain μ, and to do this, we're going to want to some sort of inverse limit of the $\Xi_{N,r}$; and if we can do this, then our strategy has worked, and will provide the p-adic L-function! Hence we've reduced the problem to the following two questions:

(1) How does one find a good integral structure on the Eisenstein differentials? That is, what is the correct (integral) definition of $\Xi_{N,r}$?

(2) How do these elements behave under the 'norm' maps

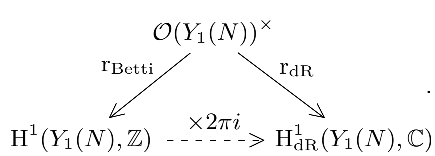
$$\mathcal{O}[(\mathbb{Z}/p^r)^\times] \longrightarrow \mathcal{O}[(\mathbb{Z}/p^r)^\times],$$

that is, what must we do to make sense of the inverse limit?

Hence we're looking for 'good integral structure and norm compatibility of Eisenstein series.' This looks an awful lot like an Euler system type of question, and indeed, the answer comes from *modular units*, which are fundamental in the construction of lots of known examples of Euler systems (for example, in David, Sarah and Antonio's construction of an Euler system for the Rankin-Selberg convolution of two modular forms).

Modular units and Eisenstein elements

Modular units (of level N) are elements of $\mathcal{O}(Y_1(N))^\times$, that is, functions on the modular curve without zeros or poles (or equivalently, rational functions on the (compact) modular curve (over $\mathbb{Q}$) whose divisors are supported on the cusps). They can be thought of as *motivic* objects and in particular we have realisation maps



Moreover, after tensoring with $\mathbb{C}$, the Betti and de Rham cohomology are isomorphic, and this isomorphism sends $\mathrm{r}_{\mathrm{Betti}}(g)$ to $2\pi i \mathrm{r}_{\mathrm{dR}}(g)$ (as given by the dotted line above).

There is a system of modular units known as *Siegel units* that is vitally important in the theory of Euler systems. Whilst we omit some of the details (in particular, the auxiliary c), the main thrust of this result is:

Theorem (Kato)**.** *For each N, there exists a modular unit $g_N \in \mathcal{O}(Y_1(N))^\times$ such that:*

(i) The g_N behave well under 'norms' from $Y_1(Nm)$ to $Y_1(N)$, and

(ii) The de Rham realisation is $\mathrm{r}_{\mathrm{dR}}(g_N) = E_N$.

Proof. See [Kat04]. □

So Siegel units given both integrality (through the isomorphism to Betti cohomology with $\mathbb{Z}$ coefficients) and norm compatibility. So we can build precise definitions of $\Xi_{N,r}$ out of these.

Definition. Let

$$\Xi_{N,r} := \sum_{a \pmod{p^r}} (\beta_{a,r})_* \iota_* \mathrm{r}_{\mathrm{Betti}}(g_N) \otimes [a] \quad \in \mathrm{H}^2(Y_1^F(\mathfrak{n}), \mathbb{Z}) \otimes \mathcal{O}[(\mathbb{Z}/p^r)^\times].$$

The main result making this work is:

Theorem (Loeffler–W.)**.** *Under the norm map*

$$\mathrm{Norm} : \mathrm{H}^2(Y_1^F(\mathfrak{n}), \mathbb{Z}) \otimes \mathcal{O}[(\mathbb{Z}/p^{r+1})^\times] \longrightarrow \mathrm{H}^2(Y_1^F(\mathfrak{n}), \mathbb{Z}) \otimes \mathcal{O}[(\mathbb{Z}/p^r)^\times]$$

which is projection in the second variable, we have

$$\mathrm{Norm}(\Xi_{N,r+1}) = (U_p)_* \Xi_{N,r},$$

where $(U_p)_*$ is the adjoint Hecke operator on the cohomology.

The p-adic Asai L-function

Combining all of the above, we've now established all of the tools to execute the strategy outlined above. In particular, we can define

$$\Xi_{N,\infty} := \varprojlim (U_p)_*^{-r} \Xi_{N,r},$$

which makes sense by the above theorem. (Strictly speaking, we should have applied a version of Hida's ordinary projector to this, since we need to invert U_p, but I'm just going to assume everything from hereon in is ordinary). Then

$$\mu(\phi) := \langle \phi, \Xi_{N,\infty} \rangle \in \mathcal{O}[[\mathbb{Z}_p^\times]]$$

defines the required distribution map, and we have:

Theorem (Loeffler–W.)**.** *Let f be an ordinary Bianchi modular form of weight 2. Then*

$$L_p^{\mathrm{As}} := \mu(\phi_f) \in \mathcal{O}[[\mathbb{Z}_p^\times]]$$

is a distribution on $\mathbb{Z}_p^\times$ interpolating the critical values of the Asai L-function, in the sense that

$$L_p^{\mathrm{As}}(f, \chi) = (*) L^{\mathrm{As}}(f, \chi, 1),$$

where $(*)$ is an explicit factor.

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