Canonical Models of Shimura Varieties

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Reference: Milne, §13, 14

Let (G,X) Shimura datum, $K \subset G(\mathbb{A}_f)$ open compact $\rightsquigarrow$ Shimura variety $\mathrm{Sh}_K(G,X)$.

Last week: Elvira defined canonical models. This week: (Sketch) proofs.

§1: Recap on canonical models

Need a number of definitions:

- Reflex field: a number field $E(G,X)$ attached to (G,X).

  (def'n: $x \in X \xleftrightarrow{\text{def'n}}$ char $h_x : \mathbb{S} \longrightarrow G_{\mathbb{R}}$
  $\rightsquigarrow$ cochar $\mu_x : \mathbb{G}_m \longrightarrow T_{\mathbb{C}}$,
  & $E(G,X)$ = field of def'n of μ_x, which is independent of x).

  Example: $G = GSp_{2n}$ is split $/\mathbb{Q}$, so $E(G,X) = \mathbb{Q}$.

- Special point: a point $x \in X$ s.t. $\exists$ torus $T_x \subset G$ s.t. $h_x : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ factors.

  $$\begin{tikzcd} & T_{x,\mathbb{R}} \arrow[dr] & \\ \mathbb{S} \arrow[ur] \arrow[rr] & & G_{\mathbb{R}} \end{tikzcd}$$

  e.g. $G = GL_2$, $X = \mathcal{H}$. Then $\tau \in \mathcal{H}$ special $\iff$ $\mathbb{Q}(\tau)/\mathbb{Q}$ is IQF.
  $\rightsquigarrow$ get $\mathbb{Q}(\tau)^\times \hookrightarrow G(\mathbb{Q})$.

  $G = GSp_{2n}$; then have

  $$\begin{tikzcd} \mathrm{Sh}_K(G,X)(\mathbb{C}) \arrow[r, leftrightarrow, "1:1"] & \{\text{ab. vars.}/\mathbb{C} + \text{structure}\}/\text{iso} \\ \{\text{special pts}\} \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow] & \{\text{CM ab. vars.} + \text{struc}\}/\text{iso}. \arrow[u, phantom, "\cup"] \end{tikzcd}$$

- Reciprocity (from arithmetic automorphisms to algebraic automorphisms)

  If (T_x, x) special pair, let $E(x) :=$ field of def'n of μ_x. Then we have homomorphisms

  $$\begin{tikzcd} \mathbb{A}_{E(x)}^\times \arrow[r, "\mu_x"] \arrow[d, "\mathrm{Art}_{E(x)}"', "(\mathrm{CFT})"] & T_x(\mathbb{A}_{E(x)}) \arrow[r, "\mathrm{Norm}"] & T(\mathbb{A}_{\mathbb{Q}}) \arrow[r, "\mathrm{proj.}"] & T(\mathbb{A}_f) \arrow[d, phantom, "\cap"] \\ \mathrm{Gal}(E(x)^{ab}/E(x)) \arrow[rrr, dashed] & & & G(\mathbb{A}_f) \end{tikzcd}$$

  "transfer arithmetic autos. to algebraic autos."

Definition: A model $M_K(G,X)$ of $\mathrm{Sh}_K(G,X)$ is <u>canonical</u> if:

- it is defined over the reflex field $E(G,X)$;
- "reciprocity": $\forall$ special points $x \in X$, and all $a \in G(\mathbb{A}_f)$, have

1) The point

$$[x,a]_K \quad \in \quad G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K \quad =: \mathrm{Sh}_K(G,X)$$

is defined over $E(x)^{ab}$, and

2) $\forall\, s \in \mathbb{A}^\times_{E(x)}$, we have

$$\mathrm{Art}_{E(x)}(s)\big([x,a]_K\big) \;=\; \big[x,\; r_x(s)\, a\big]_K .$$

Theorem: If (G,X) is a Shimura datum, then $\mathrm{Sh}_K(G,X)$ admits a unique canonical model.

§2: Descending from $\mathbb{C}$ to $E(G,X)$

General question: if $V/\mathbb{C}$ is a variety, and $E \subset \mathbb{C}$, when does V have a model over E?

First: let's go the other way. Have a base-change map

$$\{\text{Varieties}/E\} \longrightarrow \{\text{Varieties}/\mathbb{C}\},$$
$$V_0 \longmapsto V := V_0 \otimes \mathbb{C}.$$

But this is very destructive: you cannot recover V_0 from V!

↳ e.g. $\mathbb{S} \not\cong \mathbb{G}_m^2$ over $\mathbb{R}$, but both are isomorphic to $\mathbb{G}_{m,\mathbb{C}}^2$ after base-change.

... so: we need to remember more structure.

Natural way to descend to smaller fields: use "Galois action".

↳ note $\mathrm{Aut}(\mathbb{C}/E)$ fixes V_0, hence acts on $V(\mathbb{C})$ (as a <u>set</u>).

Proposition: The functor

$$\{\text{Varieties}/E\} \rightsquigarrow \{\text{Varieties } V/\mathbb{C} + \text{action } \mathrm{Aut}(\mathbb{C}/E) \circlearrowright V(\mathbb{C})\}$$

is fully faithful, i.e. can "recover" V_0 from $[V + \mathrm{Aut}(\mathbb{C}/E) \circlearrowright V(\mathbb{C})]$

... However, the RHS is <u>much</u> "bigger" than the LHS! How do we cut out the (essential) image?

We need the action of $\mathrm{Aut}(\mathbb{C}/E)$ to be "nice".

Definition: We say an action of $\mathrm{Aut}(\mathbb{C}/E) \circlearrowright V(\mathbb{C})$ is:

- regular if the action of each $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ is induced by a regular isomorphism $\sigma V \to V$ (i.e. comes from algebraic geometry),
- continuous if the action descends to some f.g. extension F/E, i.e. $\exists\, F/E$ f.g. and a model V_F/F s.t. the action of $\mathrm{Aut}(\mathbb{C}/F)$ on $V(\mathbb{C})$ is induced from V_F.

Theorem: The functor

$$\{\mathrm{Vars}/E\} \xrightarrow{\sim} \{\text{Varieties } V/\mathbb{C} + \text{regular, cts action } \mathrm{Aut}(\mathbb{C}/E) \circlearrowright V(\mathbb{C})\}$$

is an equivalence of categories.

Remark: $\exists$ a straightforward criterion for continuity: the action is continuous if $\exists$ a finite set $\{P_1, \dots, P_n\} \subset V(\mathbb{C})$ s.t.:

- (large enough) the only $\phi \in \mathrm{Aut}(V)$ s.t. $\phi(P_i) = P_i\ \forall i$ is $\phi = \mathrm{id}$,
- (def/F) if $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$, then $\sigma(P_i) = P_i \quad \forall i$.

(keep in mind: Special points!)

§3: A model for the Siegel Shimura variety

Let $(V, \psi) :=$ symplectic space of dim $2n$, $G = GSp(V,\psi) = GSp_{2n}$.

Recall: $\exists$ bijection

isogeny $A \to A^\vee$

$$Sh_K(G,X)(\mathbb{C}) \xleftrightarrow{\sim} \left\{ \begin{array}{l} (A, s, \eta K) : \ A/\mathbb{C} \text{ ab. var.}, \ s \text{ polarisation}, \\ \eta : V \otimes \mathbb{A}_f \xrightarrow{\sim} V_f A = \text{rat'l Tate module} \\ \quad (\text{compatible with } \psi \text{ and } s) \end{array} \right\}$$

$\circlearrowright$

(induced action) $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$

$\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \circlearrowright$

$(A, s, \eta K) \longmapsto (\sigma A, \sigma s, \sigma \circ \eta K)$

Proposition: The induced action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on $Sh_K(G,X)(\mathbb{C})$ is regular & continuous.

Proof: (sketch). regular: omitted. (Goes through polarised integral variation of Hodge structures).

Continuous: We use the remark.

1) Let $x \in X$. Fact: $\{[x,a]_K : a \in G(\mathbb{A}_f)\} \subset Sh_K(G,X)(\mathbb{C})$ is Zariski-dense

$\hookrightarrow$ If $\phi \in \mathrm{Aut}(Sh_K)$ with $\phi([x,a]_K) = [x,a]_K \ \forall a$, then $\phi = id$.

2) Fact: $\mathrm{Aut}(Sh_K)$ is finite $\rightsquigarrow \exists a_1, \dots, a_n \in G(\mathbb{A}_f)$ s.t. if $\phi([x,a_i]_K) = [x,a_i]_K \ \forall i$, then $\phi = id$.

3) Let x be special. Then CM theory $\rightsquigarrow \exists$ finite $F/\mathbb{Q}$ s.t. $\sigma[x,a_i]_K = [x,a_i]_K \ \forall i, \forall \sigma \in \mathrm{Aut}(\mathbb{C}/F)$

4) (2)+(3)+ Remark $\Rightarrow$ action is continuous. $\square$

Corollary: $Sh_K(G,X)$ admits a model $M_K(G,X)$ over $\mathbb{Q}$. This model is uniquely determined by the $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-action on $Sh_K(G,X)(\mathbb{C})$.

Still need to prove: reciprocity + uniqueness.

§4: Uniqueness in one paragraph

Suppose $M_K(G,X)$, $M_K'(G,X)$ are two canonical models over $E(G,X)$. Have isom

$$\tau : M_K(G,X)_{\mathbb{C}} \longrightarrow M_K'(G,X)_{\mathbb{C}}.$$

If x_0 special, then

$$\Big[\{[x_0,a]_K : a \in G(\mathbb{A}_f)\} \subset Sh_K \text{ dense} \Big] + \Big[\text{reciprocity}\Big] \Rightarrow \Big[\sigma\tau = \tau \ \forall \sigma \in \mathrm{Aut}(\mathbb{C}/E(x_0))\Big]$$

$\Rightarrow \tau$ is defined over $E(x_0)$ $\forall$ special pts

(+ some work) $\Rightarrow \tau$ is defined over $E(G,X)$

$\Rightarrow M \cong M$ over E, and canonical models are unique. $\square$

§5: Reciprocity

Recall: if (T_x, x) is special, have

$$\begin{tikzcd} \mathbb{A}_{E(x)} \arrow[r, "r_x"] \arrow[d, "\mathrm{Art}_{E(x)}"'] & T_x(\mathbb{A}_f) \subset G(\mathbb{A}_f) \\ \mathrm{Gal}(E(x)^{ab}/E(x)) & \end{tikzcd}$$

Need to prove:

$$\boxed{\begin{array}{l} \forall \text{ special points } x \in X \text{ and } \forall a \in G(\mathbb{A}_f): \\ \quad - \ [x,a]_K \in M_K(G,X)(E(x)^{ab}), \\ \quad - \ \forall s \in \mathbb{A}_{E(x)}^{\times}, \text{ have} \\ \qquad \mathrm{Art}_{E(x)}(s)\big([x,a]_K\big) = [x, r_x(s)a]_K . \end{array}}$$

<u>Proposition</u>: We have

$$\begin{array}{ccc} Sh_K(G,X) & = & \{\text{Ab vars + structure}\}/\text{iso} \\ \cup & & \\ \{\text{Special pts}\} & = & \{\underline{\underline{\text{CM}}} \text{ ab vars + struc}\}/\text{iso} \end{array}$$

<u>Key example</u>: CM tori (cf. Elvira's talk).

(E,Φ) CM type (E = CM field, $\Phi \subset \mathrm{Hom}(E,\mathbb{C})$).

$\rightsquigarrow (T, h_\Phi)$ Shimura datum, $T := \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$, h_Φ = diagonal map.

Pf of reciprocity: 1) (MODULI) $\quad Sh_K(T,h_\Phi) \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} (A,i,\eta K) : (A,i) \text{ CM ab var of type } (E,\Phi), \\ \eta K = K\text{-lvl struc} \end{array} \right\}$

2) (DESCENT) Let E^* = reflex field of (E,Φ).

CM theory $\Rightarrow \quad Sh_K(T,h_\Phi)(\mathbb{C}) = Sh_K(T,h_\Phi)(E^{*,ab})$.

3)

$$\begin{tikzcd}
\mathrm{Gal}(E^{*,ab}/E^{ab}) \arrow[d, equal] & & \mathrm{Gal}(E^{*,ab}/E^*) \arrow[ll, "\text{def'n}"'] \arrow[d, phantom, "\begin{array}{c}\text{Main thm of CM} \\ \text{(Muhammad)}\end{array}"] \\
\text{alg. action of } r_x & Sh_K(T,h_\Phi)(\mathbb{C}) = \{(A,i,\eta K)/E^{*,ab}\} \arrow[d, equal] & \text{action via isogenies + reflex norm } N_{\Phi^*} \arrow[ll, "(\text{Elvira})", "r_x = N_{\Phi^*}"'] \\
& \{\text{special pt of type } (E,\Phi)\} & \\
& \cap & \\
& Sh_K(G,X)(\mathbb{C}) &
\end{tikzcd}$$

4) Going round the diagram $\Rightarrow$ reciprocity for $Sh_K(T,h_\Phi) \Rightarrow$ for $\big[\text{type } (E,\Phi)\text{-special pts}\big] \subset Sh_K(G,X)$

5) General special pts ($\hookrightarrow$ ab. vars. with CM by some CM algebra, not necessarily a field): similar.