p-ADIC L-FUNCTIONS FOR SYMPLECTIC REPRESENTATIONS OF GL(2n)

Nottingham, 12th Feb 2020

Chris Williams, joint with Daniel Barrera & Mladen Dimitrov

§0: Motivation

Let π automorphic rep'n, $L(\pi, s)$ attached L-fn. Bloch-Kato conjecture:

$\boxed{\text{Arithmetic invariants of } \pi} \longleftrightarrow \boxed{\text{Special values of } L(\pi,s)}$

... generalises BSD, class no. formula: extremely difficult.

Iwasawa Main Conjectures: p-adic reformulation

$\boxed{\text{p-adic invariants of } \pi} \longleftrightarrow \boxed{\text{p-adic L-function } L_p(\pi)}$

$\rightarrow$ more tractable, important consequences for classical Bloch-Kato.

However, to <u>state</u> an IMC...

... need to prove $L_p(\pi)$ exists!

<u>Set-up:</u> $G = GL_n$, π automorphic rep'n of $G(\mathbb{A})$, $L(\pi,s)$ standard L-fn of π.

$\rightsquigarrow \Lambda(\pi,s)$ completed at ∞.

Expect:

(1) [algebraicity] $\exists\, J \subset \mathbb{Z}$ "critical integers" and $\Omega_\pi^\pm \in \mathbb{C}^\times$ s.t.

$$\Lambda(\pi,\chi,j+1)/\Omega_\pi^\pm \in \overline{\mathbb{Q}} \qquad (\forall \chi \text{ Dirichlet}, \forall j \in J).$$

(2) [p-adic L-fn] Can p-adically interpolate (1).

$\left(\text{Notation: } \mathcal{A} = \{\text{locally analytic fns } \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times\} \ni \left(\varphi_{\chi,j} : x \mapsto \chi(x)x^j\right) : \text{cond}(\chi) = p^n, j \in \mathbb{Z} \right.$

$\left. \mathcal{D} = \mathcal{A}^\vee. \right)$

$\rightsquigarrow$ expect $\exists\, L_p^\pi \in \mathcal{D}$ s.t. $\forall j \in J$, χ cond. p^n, have

$$L_p(\pi,\chi,j) := L_p^\pi(\varphi_{\chi,j}) = \int_{\mathbb{Z}_p^\times} \chi(x)x^j \cdot L_p^\pi = (*)\,\Lambda(\pi,\chi,j+1)/\Omega_\pi^\pm.$$

$\uparrow$ explicit

L_p^π = <u>p-adic L-function</u> of π (+ growth condition that <u>usually</u> $\Rightarrow L_p^\pi$ unique).

(3) L_p^π varies analytically in p-adic weight families.

What is known?

- $n=1$: ✓
- $n=2$: (✓)
- $n \geqslant 3$: poorly understood.

<u>Today</u>: case $GL(2n)$, π "symplectic" cohomological.

- π p-ordinary: constructions of L_p^π by Gehrmann, Dimitrov-Januszewski-Raghuram.
- today: new construction via overconvergent cohomology;
  - non-ordinary π,
  - (in progress) variation in families.

§1: Set-up + main theorem

Notation: $G = GL_{2n} \supset H = GL_n \times GL_n$,

$$J_p = \left\{ g \in G(\mathbb{Z}_p) : g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p \right\} \qquad \text{"parahoric"}$$

π auto. rep'n of $G(\mathbb{A})$: (cuspidal, regular, algebraic)

- spherical/unramified at p $\quad (\pi_p^{G(\mathbb{Z}_p)} \neq 0)$,
- dominant, cohomological wt
$$\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{2n}),$$
- L-fn $L(\pi, s)$,
- Critical integers: $J = \{ j : \lambda_n \geqslant j \geqslant \lambda_{n+1} \}$.

Let

$$U_p \sim \begin{pmatrix} pI_n & \\ & I_n \end{pmatrix} \qquad \text{Hecke operator},$$

$$\alpha = \text{simple eigenvalue of } U_p \circlearrowleft \pi_p^{J_p}.$$

$$\rightsquigarrow \quad \tilde{\pi} = (\pi, \alpha) \quad \text{"}U_p\text{-stabilisation"}. \qquad \left(\text{note: } \pi \text{ ordinary} \iff v_p(\alpha) = 0\right).$$

Example: $n=1$ $(GL(2))$;

$$\pi \longleftrightarrow \text{modular eigenform},$$
$$\lambda = (k, 0) \longleftrightarrow \text{wt } k+2,$$

Critical values $L(f, \chi, j+1)$: $0 \leq j \leq k$.

ASSUMPTION: π is <u>symplectic</u>, i.e. functorial transfer of Π of $GSpin_{2n+1}(\mathbb{A})$

on Galois rep'ns:

$$\begin{tikzcd} \rho_\Pi : G_{\mathbb{Q}} \arrow[r] \arrow[rr, bend right, "\rho_\pi"'] & GSp_{2n} \subset & GL_{2n}. \end{tikzcd}$$

- $GL(2)$: always true.
- $GL(4)$: $L(\pi, s)$ = L-fn of genus 2 Siegel modular form.

$$\boxed{\text{Theorem: (BDW). Suppose } v_p(\alpha) < \lambda_n - \lambda_{n+1} + 1. \text{ Then } L_p^\pi \text{ exists and is unique.}}$$

§2: Classical cohomology (blue on diagram below)

Aim: relate critical L-values to cohomology groups

$$H_c^t(S_G, V_\lambda^\vee),$$

where:

- $t = n^2 - n + 1$ (top degree),
- S_G = locally symmetric space for G, level K, $K_p \subset J_p$,
- $V_\lambda = \mathrm{Ind}_{B^-}^G \lambda$
  = concretely: functions on $\begin{pmatrix} 1 & x & y & \cdots & v \\ & \ddots & z & & \vdots \\ & & \ddots & & w \\ & & & & 1 \end{pmatrix}$, polynomial in $x, y, z, \ldots, w$, degree dep. on λ.

Example: $GL(2)$; - $t = 1$,
- $S_G = Y_1(N)$ modular curve, $p \mid N$,
- V_λ = polynomials of deg $\leq k$,
- $H_c^1(Y_1(N), V_\lambda^\vee)$ = modular symbols.

<u>Fact</u>: $\exists$ "nice" class

$$\phi_{\tilde{\pi}} \in H_c^t(S_G, V_\lambda^\vee) \qquad \text{(same Hecke eigenvalues!)}$$

<u>Proposition</u>: (Grobner-Raghuram, Dimitrov-Januszewski-Raghuram). Let $j \in J$, χ conductor p^r, $r \geq 1$. Then $\exists$ map

$$Ev_{\chi,j} : H_c^t(S_G, V_\lambda^\vee) \longrightarrow \overline{\mathbb{Q}}$$

such that

$$Ev_{\chi,j}(\phi_\pi) = (*) \, \Lambda(\pi, \chi, j+1) / \Omega_\pi^\pm .$$

Idea of proof: - Friedberg-Jacquet: π symplectic $\rightsquigarrow$ adelic integral formula for $L(\pi, s)$ (over H)
- Define "automorphic cycles" $X_r \subset S_G$ (locally symmetric spaces for H, dimension t)
- pull back to X_r, integrate, relate to F-J integral. ($\square$)

Picture:

§3 AIM:

$$\begin{tikzcd}
H^t_c(S_G, \mathcal{D}_\lambda) \ni \Psi_{\tilde\pi} \arrow[rr, "Ev"] \arrow[d, "\rho"'] & & L_p^{\tilde\pi} \arrow[d, mapsto] & \in & \mathcal{D} \arrow[d, "\int \chi(x) x^j"] \\
H^t_c(S_G, V_\lambda^\vee) \ni \phi_{\tilde\pi} \arrow[u, dashed, bend right, "v_p(\alpha) < \lambda_n - \lambda_{n+1} + 1"'] \arrow[rr, "Ev_{\chi,j}"] & & \Lambda(\pi, \chi, j+1)/\Omega^\pm_{\tilde\pi} & \in & \overline{\mathbb{Q}}_p \\
& \tilde\pi \arrow[ul, bend left] \arrow[ur, bend right] & & &
\end{tikzcd}$$

symplectic, cohomological

§2

§3: Overconvergent cohomology

Idea: interpolate $V_\lambda^\vee$.

Generally: "$\mathrm{Ind}_{B^-}^G$" $\sim$ functions on $\begin{pmatrix} 1 & x & y & \cdots & v \\ & \ddots & ? & & \vdots \\ & & \ddots & & w \\ & & & & 1 \end{pmatrix}$.

Coefficients:

Classical	V_λ	$\mathrm{Ind}_{B^-}^G \lambda$	$\begin{pmatrix} 1 & & \text{poly} \\ & \ddots & \\ & & 1 \end{pmatrix}$	$V_\lambda^\vee$
	$\cap$			$\uparrow$
"partially overconvergent"	$\mathcal{A}_\lambda$	$\mathrm{LA}\text{-}\mathrm{Ind}_{Q \cap J_p}^{J_p} \mathrm{Ind}_{B^- \cap H}^{H} \lambda$	$\left(\begin{array}{c\|c} \begin{smallmatrix} 1 & \text{poly} \\ & 1 \end{smallmatrix} & \text{an} \\ \hline & \begin{smallmatrix} 1 & \text{poly} \\ & 1 \end{smallmatrix} \end{array}\right)$	$\mathcal{D}_\lambda$
	$\cap$			$\uparrow$
overconvergent (shot too far!)	$\mathcal{A}_\lambda^{\text{full}}$	$\mathrm{LA}\text{-}\mathrm{Ind}_{B^- \cap I_p}^{I_p} \lambda$	$\begin{pmatrix} 1 & & \text{an} \\ & \ddots & \\ & & 1 \end{pmatrix}$	$\mathcal{D}_\lambda^{\text{full}}$

$\longrightarrow$ get induced map

$$\rho : H^t_c(S_G, \mathcal{D}_\lambda) \longrightarrow H^t_c(S_G, V_\lambda^\vee).$$

Theorem: (BDW, Urban). If $v_p(\alpha) < \lambda_n - \lambda_{n+1} + 1$, then the restriction

$$\rho : H^t_c(S_G, \mathcal{D}_\lambda)^{U_p = \alpha} \longrightarrow H^t_c(S_G, V_\lambda^\vee)^{U_p = \alpha}$$

is an isomorphism.

on diagram above

<u>Corollary</u>: $H^t_c(S_G, \mathcal{D}_\lambda)_\pi \cong H^t_c(S_G, V_\lambda^\vee)$

$$\Psi_{\tilde{\pi}} \longleftarrow\!\dashrightarrow \phi_{\tilde{\pi}}.$$

<u>Theorem</u>: (BDW). $\exists$ map

$$\mathrm{Ev} : H^t_c(S_G, \mathcal{D}_\lambda)_\pi \longrightarrow \mathcal{D}$$

interpolating $\mathrm{Ev}_{\chi,j}$; ie. for all $\chi, j \in J$, have

$$\varphi_{\chi,j} \circ \mathrm{Ev} = \mathrm{Ev}_{\chi,j} \circ \rho.$$

* subtlety: really Ev_{p^r} $\forall r$, and "$\mathrm{Ev} = U_p^{-r} \mathrm{Ev}_{p^r}$". Only makes sense when U_p invertible. (small slope)

<u>Proof</u>: use same automorphic cycles X_r.

<u>Conclusion</u>: $L_p^{\tilde{\pi}} := \mathrm{Ev}(\Psi_{\tilde{\pi}}) \in \mathcal{D}$

$= p$-adic L-function of $\tilde{\pi}$.

in diagram above