Classical Symplectic Families for GL(2n)

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§1: Congruences for modular forms

Let p prime, N prime to p, $k \in \mathbb{N}$,

$$f(q) = \sum_{n \geq 0} a_n q^n \in M_k(\Gamma_0(Np)) \quad \text{(normalized) eigenform.}$$

Question: For $m \geq 1$, does there exist an eigenform

$$g(q) = \sum_{n \geq 0} b_n q^n \in M_\ell(\Gamma_0(Np))$$

such that

$$f \equiv g \pmod{p^m} \iff \forall n, \quad a_n \equiv b_n \pmod{p^m} \ ?$$

Example: (p-adic properties of Eisenstein series). $k \geq 4$ even, $N = 1$,

$$E_k(q) := \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n \in M_k(SL_2(\mathbb{Z})),$$

$$\sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1}.$$

Let

$$\tilde{E}_k(q) := E_k(q) - p^{k-1} E_k(q^p) \in M_k(\Gamma_0(p))$$

"p-refinement"

$$= \frac{(1-p^{k-1})\zeta(1-k)}{2} + \sum_{n \geq 1} \tilde{\sigma}_{k-1}(n) q^n,$$

$$\tilde{\sigma}_{k-1}(n) = \sum_{\substack{d \mid n \\ (p,d)=1}} d^{k-1}.$$

If $k \equiv \ell \pmod{p-1}$, then:

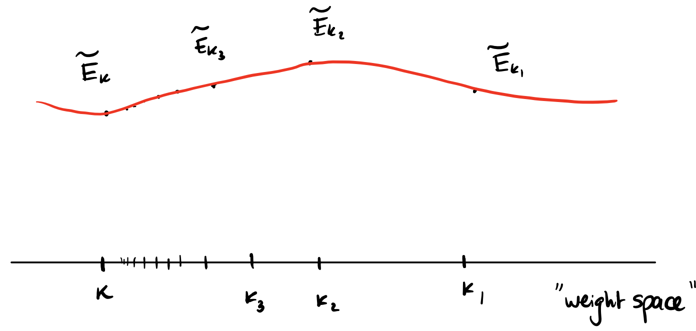
$$\left.\begin{array}{l} - \text{ FLT: } \tilde{\sigma}_{k-1}(n) \equiv \tilde{\sigma}_{\ell-1}(n) \pmod{p} \quad \forall n, \\ - \text{ Kummer: } (1-p^{k-1})\zeta(1-k) \equiv (1-p^{\ell-1})\zeta(1-\ell) \pmod{p} \end{array}\right\} \Rightarrow \tilde{E}_k \equiv \tilde{E}_\ell \pmod{p}.$$

Same proof shows:

$$\left[k \equiv \ell \pmod{\phi(p^m)}\right] \Rightarrow \left[\tilde{E}_k \equiv \tilde{E}_\ell \pmod{p^m}\right].$$

Theorem: Let $(k_m) \subset \mathbb{Z}$ with $k_m \to k$ p-adically. Then

$$\tilde{E}_{k_m} \longrightarrow \tilde{E}_k \quad p\text{-adically.}$$



Thus: the Eisenstein series $\widetilde{E}_k$ can be interpolated into a 1-dimensional p-adic family.

Amazing thing: you can also do this for cusp forms!

More precisely:
- $f \in S_k(\Gamma_0(N))$ newform,
- $\tilde{f} \in S_k(\Gamma_0(Np))$ p-refinement.

There are 2 choices of $\tilde{f}$: $\begin{cases} f(q) - \alpha f(q^p), \\ f(q) - \beta f(q^p), \end{cases}$ where $(X-\alpha)(X-\beta) = X^2 - a_p(f) + p^{k-1}$.

Theorem: (Hida, Coleman–Mazur). There is a 1-dimensional classical p-adic family through $\tilde{f}$.

§2: The eigencurve

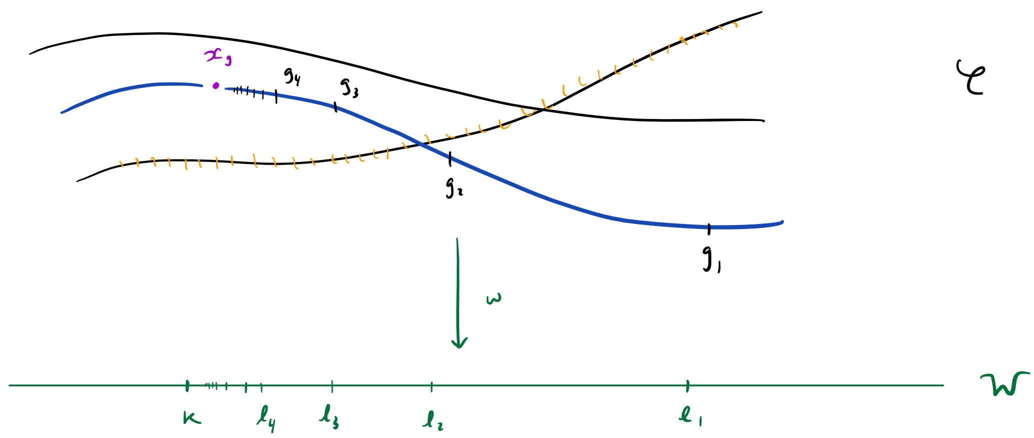
Better approach: $\{\text{eigenforms}\} \longleftrightarrow \{\text{systems of Hecke eigenvalues}\}$.

Definition: The Hecke algebra is $\mathcal{H} := \mathbb{Q}_p[\{T_\ell : \ell \nmid Np\}, U_p]$.

If g is an eigenform, we have an attached eigensystem
$$\phi_g : \mathcal{H} \longrightarrow \overline{\mathbb{Q}}_p$$
such that
$$Tg = \phi_g(T)g \quad \forall\, T \in \mathcal{H}.$$

(e.g. $T_\ell \mapsto a_\ell(g)$; send T to its eigenvalue on g)

Theorem: (Coleman–Mazur) There exists an eigencurve, a moduli space for systems $\phi : \mathcal{H} \longrightarrow \overline{\mathbb{Q}}_p$ appearing in (p-adic) modular forms of level Np.



<u>Properties:</u> 1) $\exists$ weight space $\mathcal{W}$ = 1-dim rigid analytic space

$$\mathcal{W} \supset \mathbb{Z} = \text{"classical weights"},$$

and a weight map $w : \mathscr{C} \longrightarrow \mathcal{W}$.

2) $\tilde{f} \in S_\kappa(\Gamma_0(Np)) \rightsquigarrow \phi_{\tilde{f}} \rightsquigarrow x_{\tilde{f}} \in \mathscr{C}$, a classical point

3) Classical family: subspace where classical points are Zariski-dense.

<u>Theorem:</u> (Coleman–Mazur). $\exists$ 1-dimensional classical family through any $x_{\tilde{f}}$.

<u>Corollary:</u> $\forall m \geq 1$, $\exists \ell_m \geq 1$ and eigenforms $g_m \in S_{\ell_m}(\Gamma_0(Np))$ such that

$$\tilde{f} \equiv g_m \pmod{p^m}, \qquad \ell_m \longrightarrow \kappa.$$

§3: Generalisations

(modular forms are automorphic forms for $GL_2/\mathbb{Q}$).

More generally:

$GL_2/\mathbb{Q} \rightsquigarrow$ reductive group G,

$f \rightsquigarrow$ cuspidal automorphic rep'n π of $G(\mathbb{A})$,

$\phi_{\tilde{f}} \rightsquigarrow$ eigensystem $\phi_{\tilde{\pi}}$ appearing in π^{Iw} (a "classical cuspidal eigensystem").

<u>Question:</u> Does every classical cuspidal eigensystem vary in a classical p-adic family?

Two cases:

(A) – $G(\mathbb{R})$ admits discrete series
– cusp forms appear in 1 degree of cohomology
– expect every $\phi_{\tilde{\pi}}$ varies in classical family

} e.g. Hilbert mod forms, Siegel mod forms, any G admitting a Shimura variety

(B) – No discrete series
– cusp forms in multiple degrees
– expect $\exists \phi_{\tilde{\pi}}$ that don't vary in classical family

} eg. Bianchi mod forms, GL_n for $n \geq 3$

Folklore expectation: "every classical family comes from case A".

↳ In case (A), there are always systematic congruences between eigensystems.
In case (B), there are no new systematic congruences: all such congruences are induced from a case (A) setting.

Conjectures: (1) (Calegari–Mazur) Every classical cuspidal Bianchi family is either:
– twisted base-change transfer from $GL_2/\mathbb{Q}$,
– CM transfer from GL_1/K, K/F quadratic.

(2) (Ash–Pollack–Stevens) Every classical cuspidal family for GL_3 is symmetric square transfer from $GL_2/\mathbb{Q}$.

⟶ not seeing any "new" congruences, in either case.

§4: Symplectic families

(we treat GL_{2n}. I will specialise to GL_4).

Let π = cohomological cuspidal automorphic representation of $GL_4(\mathbb{A})$. Assume:
– π_p is unramified and regular,
– π is symplectic $\iff$ transfer of some Π on GSp_4.

There are 24 p-refinements $\tilde{\pi}$ of π $\quad$ ($\leftrightarrow \dim \pi_p^{Iw} = 24$).

Conjecture: (Barrera–Graham–W.) Of the 24 p-refinements,
– 8 vary in a 2-dimensional classical family,
– 8 vary in a 1-dimensional classical family,
– 8 vary in no classical family.

Theorem: (BGW).
(1) Any symplectic classical family through $\tilde{\pi}$ has at <u>most</u> the conjectured dimension.
(2) If $\tilde{\pi}$ has non-critical slope, $\exists$ a symplectic classical family through $\tilde{\pi}$ of exactly the conjectured dimension.

(1) $\Rightarrow$ there are no "new" systematic congruences between eigensystems in symplectic representations.

<u>Key idea:</u> Refinements are stratified by parabolic subgroups $P \subset G$. Let
$$J_P := \{ g \in GL_4(\mathbb{Z}_p) : g \ (\mathrm{mod}\ p) \in Q(\mathbb{F}_p) \}.$$

There are 3 interesting parabolics:

"unlike GL(2), can find interesting eigensystems in <u>between</u> B & G"

$B = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \rightsquigarrow J_B = Iw \qquad \pi_p^{Iw} = 24\text{-dim} \longrightarrow 24 \ B\text{-refinements}$

$\cap \qquad\qquad \cap \qquad\qquad \cup$

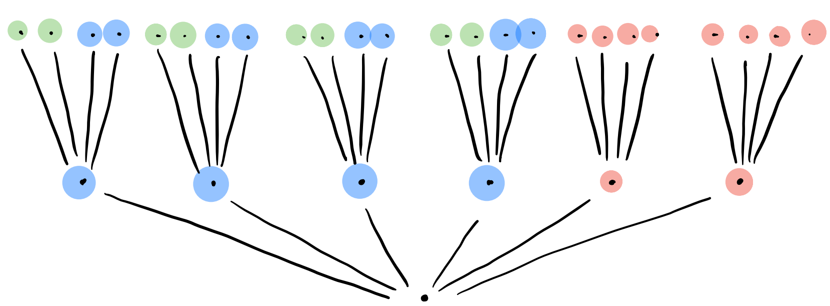
$Q = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \rightsquigarrow J_Q \qquad \pi_p^{J_Q} = 6\text{-dim} \longrightarrow 6 \ Q\text{-refinements}$

$\cap \qquad\qquad \cap \qquad\qquad \cup$

$G = (\,*\,) \rightsquigarrow J_G = GL_4(\mathbb{Z}_p) \qquad \pi_p^{J_G} = 1\text{-dim} \rightsquigarrow 1 \ G\text{-refinement}$

π_p^{Iw} (24)

$\pi_p^{J_Q}$ (6)



$\pi_p^{J_G}$ (1)

<u>For GSp_4:</u>

$B' = (\ ) \rightsquigarrow \pi_p^{J_{B'}} = 8\text{-dim} \rightsquigarrow 8 \ B'\text{-refs, varying in 2-dim families}$

$\cap \qquad\qquad \cup$

$Q' = (\ ) \longrightarrow \pi_p^{J_{Q'}} = 4\text{-dim} \rightsquigarrow 4 \ Q'\text{-refs, varying in 1-dim fams}$

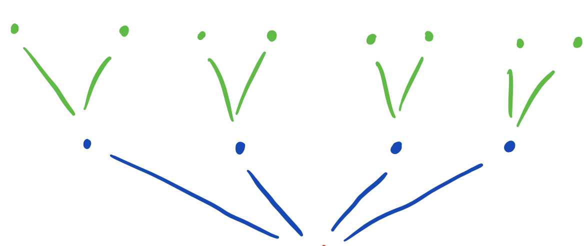
$\cap \qquad\qquad \cup$

$G' = (\ ) \rightsquigarrow \pi_p^{J_{G'}} = 1\text{-dim}$

B'

Q'

G'



For every $\tilde{\pi}$, $\exists!$ minimal $P \subset GL_4$ s.t. "$\tilde{\pi}$ is the transfer of a P'-refinement of π":
8 for B, 8 for Q, and 8 for G.