Adic Spaces

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Recap: K non-archimedean field. We want a "good" theory of analytic geometry over K.

Desirable properties:
1) "GAGA";
2) Integral models ("analytic geometry over $\mathcal{O}_K$").

Successful theories:
1) rigid spaces;
2) formal schemes.

§1: GAGA

Serre: $\exists$ functor

$$\left\{\begin{matrix}\text{schemes loc. of} \\ \text{finite type } /\mathbb{C}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Complex analytic} \\ \text{spaces}\end{matrix}\right\}$$

$$X \longmapsto X^{an},$$

(when X proper)
+ equivalence $\{\text{coherent sheaves on } X\} \simeq \{\text{coherent sheaves on } X^{an}\}$.

(via "allowable functions"): $\rightsquigarrow$ crucially: can recreate X from X^{an}.

(basic construction: closed pts of affine piece are subsets of $\mathbb{C}^n$).

$\rightsquigarrow$ can use techniques from complex analysis / diff geometry in alg. geometry (and vice versa).

Example: $\Gamma \leq SL_2(\mathbb{Z})$, modular curve $Y^{an}_{\Gamma,\mathbb{C}} := \Gamma\backslash\mathcal{H}$ = complex analytic curve

modular forms

p-adic modular forms?

$$\begin{tikzcd}
(\mathbb{C}\text{-analytic}) & Y^{an}_{\Gamma,\mathbb{C}} \arrow[d, leftrightarrow, "\text{GAGA}"] & & Y^{an}_{\Gamma,\mathbb{Q}_p} & \\
(\mathbb{C}\text{-algebraic}) & Y_{\Gamma,\mathbb{C}} & & Y_{\Gamma,\mathbb{Q}_p} \arrow[u, dashed] & := Y_{\Gamma,\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{Q}_p \\
& & Y_{\Gamma,\mathbb{Q}} \arrow[ul, leftrightarrow] \arrow[ur, leftarrow] & &
\end{tikzcd}$$

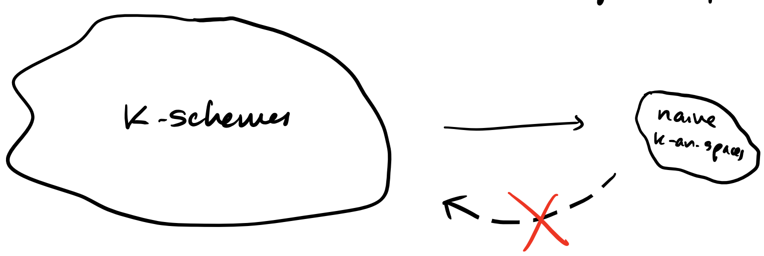
($\mathbb{Q}$-algebraic)

So: want GAGA for analytic spaces over K.

Already seen: "obvious" analogue is very bad.

(non-arch topology $\rightsquigarrow$ totally disconnected $\rightsquigarrow$ too many "locally analytic" functions (allowable)
$\rightsquigarrow$ too many isomorphisms
$\rightsquigarrow$ not enough isomorphism classes!

eg.



§2: Rigid analytic spaces

Chris' talk: introduced rigid analytic spaces.

Recall: Local models: $\max\mathrm{Spec}(A)$, $\quad A = \dfrac{K\langle z_1, \dots, z_n\rangle}{(f_1, \dots, f_r)}$ affinoid algebra.

Chris' talk: big problems with gluing.
$\rightsquigarrow$ must use fiddly notion of G-topology:
"only allow specific kind of open sets/coverings".

Key example: $X = \max\mathrm{Spec}(\mathbb{Q}_p\langle T\rangle)$ = closed rigid disc;
$X(K) = \{x \in K : |x| \leq 1\}$.

$Y = \max\mathrm{Spec}(\mathbb{Q}_p\langle T, T^{-1}\rangle)$ = unit circle;
$X(K) = \{x \in K : |x| = 1\}$.

Z = open rigid disc;
$Z(K) = \{x \in K : |x| < 1\}$. NOT a local model, but infinite union of local models.

In p-adic topology: $X = Y \cup Z$ disconnected.

In rigid topology: Y, Z admissible open sets in X, but $Y \cup Z$ not admissible covering.
$\hookrightarrow$ X is connected!

However...

> **Theorem**: $\exists$ functor
>
> $$\left\{\begin{matrix}\text{projective schemes}\\ \text{over } K\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{rigid analytic}\\ \text{spaces } /K\end{matrix}\right\},$$
>
> $$X \longmapsto X^{an},$$
>
> (Kopf?)
>
> \+ equivalence of coherent sheaves.

↳ can build wonderful theory of p-adic (+ overconvergent) modular forms out of rigid analytic modular curves.

Examples: 1) $E/\mathbb{Q}$ elliptic curve. Attached to $E/\mathbb{Q}_p$ have rigid curve $\mathcal{E}/\mathbb{Q}_p$.

Theorem (Tate). Suppose E has multiplicative reduction at p. Then $\exists q \in \mathbb{Q}_p^\times$ s.t.

$$\mathcal{E} \cong \mathbb{G}_m / q^{\mathbb{Z}}$$

over a quadratic extn of $\mathbb{Q}_p$.

$\rightsquigarrow$ "$\frac{1}{2}$ 'od'" q has deep arithmetic interpretations via $\mathcal{L}$-invariants,

(p-adic Hodge theory, exceptional zeros, Iwasawa theory)

2) Rigid analytic modular curve $\xrightarrow{\text{Coleman}}$ p-adic modular forms (eigencurve, p-adic families).

Upshot: Whilst fiddly to define, rigid analytic spaces have been very successful.

Remarks: (1) For objects "loc. of finite type $/K$", rigid spaces & adic spaces are essentially the same. (equivalence of categories)

(2) Historically, rigid spaces have been easier to work with (more concrete, closer to classical geometry)

(3) ... However: adic spaces have more intuitive geometric properties; e.g. in example above,

(rigid) $\exists$ closed immersion $\underset{\text{open disc}}{Y} \sqcup \underset{\text{circle}}{Z} \overset{\iota}{\hookrightarrow} \underset{\text{closed disc}}{X}$, $(Y \sqcup Z)(K) = X(K)$ on pts, ι not isomorphism

(G-topology, threw out this cover)

(adic) $\exists$ extra points! $\exists$ pt $g \in X(K)$ "between $|x| = 1$ and $|x| > 1$"

$\rightsquigarrow$ $g \notin Y(K), Z(K)$, $Y \sqcup Z$ not cover (in regular topology)

(+ sheaf property: Chris' talk)

§2: Integral models: formal schemes

Have $\operatorname{Spec} \mathbb{Z}_p = \{(0), (p)\}$

$$\begin{tikzcd} & \{(0), (p)\} & \\ \operatorname{Spec}(\mathbb{Q}_p) \arrow[ur] & & \operatorname{Spec}(\mathbb{F}_p) \arrow[ul] \end{tikzcd}$$

Let $X_{\mathbb{Z}_p}$ scheme over $\mathbb{Z}_p$,

$X_{\mathbb{Q}_p} = X_{\mathbb{Z}_p} \times \operatorname{Spec} \mathbb{Q}_p$,

$X_{\mathbb{F}_p} = X_{\mathbb{Z}_p} \times \operatorname{Spec} \mathbb{F}_p$.

$$\begin{tikzcd} X_{\mathbb{Q}_p} \arrow[rr, squiggly] & & X^{an}_{\mathbb{Q}_p} & \\ X_{\mathbb{Z}_p} \arrow[u, "\text{gen. fibre}"] \arrow[dr, "\text{spec. fibre}"] & & X^{an}_{\mathbb{Z}_p}\ ? \arrow[u] \arrow[dr] & \arrow[l, squiggly] \text{answer for rigid spaces: } \underline{\text{formal schemes}}. \\ & X_{\mathbb{F}_p} & & X^{an}_{\mathbb{F}_p}\ ? \end{tikzcd}$$

Let A = commutative topological ring, e.g. $\mathbb{C}, \mathbb{Q}_p, \mathbb{Z}_p, \mathbb{Z}_p[[T]]$.

Observation: $[\operatorname{Spec}(A) + \text{Zariski topology}]$ does not "see" the topology on A.

Formal schemes: refinement for special class of topological rings, "adic rings".

("rigid spaces: use topology on K to refine allowable functions. Formal schemes: use topology on rings to refine local models")

<u>Def'n</u>: A commutative ring, $I \subset A$ ideal. The I-adic topology is the topology where

$$\{I^n : n \geq 0\} = \text{fundamental basis of nhbds of } 0,$$

ie.

$$\text{Subset } X \subset A \text{ is open} \iff X = \text{union of cosets } a + I^n.$$

A topological ring A is <u>adic</u> if $\exists$ ideal $I \subset A$ s.t. topology is the I-adic topology.

↳ say I is an ideal of definition.

eg. - $\mathbb{C}, \mathbb{Q}_p$ <u>not</u> adic rings with usual topologies.

- $\mathbb{Z}_p$ is an adic ring, $I = (p)$.
- $\mathbb{Z}_p[[T]]$, $I = (p, T)$.
- <u>any</u> A with discrete topology, $I = (0)$.

Def'n: (Formal scheme) Let A be an I-adic ring. Define the formal spectrum of A to be

$$\operatorname{Spf} A := \{ \mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{p} \text{ open} \},$$

+ gluing dep. on topology

Basis of open sets

$$D(f) := \{ \mathfrak{p} \in \operatorname{Spf} A : f \notin \mathfrak{p} \} \quad \text{for } f \in A,$$

Structure sheaf

$$\mathcal{O}_{\operatorname{Spf} A}(D(f)) = I\text{-adic completion of } A[f^{-1}]$$
$$:= \varprojlim_n A[f^{-1}]/I^n.$$

A formal scheme is a topologically ringed space locally of form $\operatorname{Spf} A$ for an adic ring A.

⟶ we remember the topologies on A!

e.g. - $\operatorname{Spf}(\mathbb{Z}_p) = \{ (p) \}$.

- $X = \operatorname{Spf}(\mathbb{Z}_p[[T]])$ = formal open disc over $\mathbb{Z}_p$; if $K/\mathbb{Q}_p$ non-arch, $\mathcal{O}_K$ = ring of integers.

Then $X(\mathcal{O}_K) = \mathfrak{m}_K$ max ideal.

- Every scheme is a formal scheme: eg. $\operatorname{Spec} A = \operatorname{Spf}(A, \text{discrete top})$.

(genuinely enlarged our working space).

Raynaud: Formal schemes over $\operatorname{Spf} \mathbb{Z}_p$, p-adic topology, give "good" integral non-arch. geometry.

PROBLEM: $\mathbb{Q}_p$ w/ p-adic topology is not adic! ... ⟶ $\operatorname{Spf} \mathbb{Q}_p$ doesn't make sense;

⟶ no obvious "generic fibre"!

Theorem: (Berthelot). $\exists$ a "generic fibre functor"

$$\left\{ \begin{matrix} \text{locally finite type} \\ \text{formal schemes} / \operatorname{Spf} \mathbb{Z}_p \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{rigid analytic} \\ \text{spaces} / \mathbb{Q}_p \end{matrix} \right\}$$

$$X \longmapsto X_\eta .$$

<u>Remarks:</u> 1) Further evidence for utility of rigid spaces: formal schemes occur very naturally.

2) Construction is very involved...

3) local models are <u>not</u> sent to local models!!

e.g: formal open unit disc over $\mathbb{Z}_p$ is $X = \mathrm{Spf}\, \mathbb{Z}_p[\![T]\!]$.

Fact: X_η = rigid open unit disc over $\mathbb{Q}_p$.

$$\ldots = \bigcup_{n \geq 1} \mathcal{U}\left(\frac{Z^n}{p}\right)$$

$\neq \mathrm{maxSpec}(A)$ for any A!!

§3: Adic reformulation

In world of adic spaces, recover picture

$$\text{"}\begin{tikzcd} & \mathrm{Spa}(\mathbb{Z}_p) & \\ \mathrm{Spa}(\mathbb{F}_p) \arrow[ur] & & \mathrm{Spa}(\mathbb{Q}_p) \arrow[ul] \end{tikzcd}\text{"}$$

If X adic $/\mathrm{Spa}(\mathbb{Z}_p)$, can define honest generic fibre $X_\eta := X \times_{\mathrm{Spa}(\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p)$

and have:

$$\begin{tikzcd} \{\text{Formal schemes} / \mathrm{Spf}\,\mathbb{Z}_p\} \arrow[r, "X \longmapsto X_\eta"] \arrow[d, "X \mapsto X^{ad}"] & \{\text{rigid spaces} / \mathbb{Q}_p\} \arrow[d, "X \mapsto X^{ad}"] \\ \{\text{Adic spaces} / \mathrm{Spa}(\mathbb{Z}_p)\} \arrow[r, "X^{ad} \longmapsto X^{ad}_\eta"] & \{\text{adic spaces} / \mathrm{Spa}(\mathbb{Q}_p)\}. \end{tikzcd}$$

A little more on G-topologies

Def'n: Let $x \in \mathrm{MaxSpec}(A)$. Then A/x = finite extension of K

$\hookrightarrow \exists!$ extension of $|\cdot|$ to A/x.

If $f \in A$, define $f(x) :=$ image of f in A/x,
$|f(x)|$ its valuation.

Def'n: Let $f_1, \dots, f_r, g \in A$. Define the rational domain

$$\mathcal{U}\left(\frac{f_1, \dots, f_r}{g}\right) := \left\{ x \in A : |f_i(x)| \leq |g(x)| \ \forall i \right\}$$

$$= \mathrm{maxSpec}\left(A\langle T_1, \dots, T_r \rangle / (gT_i - f_i) \right).$$

In the G-topology on $\mathrm{MaxSpec}(A)$: open sets are built from rational domains.

Example: shortcomings of G-topologies

eg. 1) $A = \mathbb{Q}_p\langle Z \rangle$, $X = \mathrm{MaxSpec}(A)$. Then X = closed rigid disc over $\mathbb{Q}_p$,
ie. if $K/\mathbb{Q}_p$, then $X(K) = \mathcal{O}_K = \{ x \in K : |x| \leq 1 \}$.

Have rational domains

$$\mathcal{U}\left(\frac{Z^n}{p}\right) = \left\{ x \in K : |x| \leq p^{-1/n} \right\},$$

and

$$V = \bigcup_{n \geq 1} \mathcal{U}\left(\frac{Z^n}{p}\right) = \{ x \in K : |x| < 1 \}$$

rigid open unit disc.

2) $B = \mathbb{Q}_p\langle Z, Z^{-1} \rangle$, $Y = \mathrm{maxSpec}(B)$ = unit circle,

$$Y(K) = \{ x \in K : |x| = 1 \}.$$

Observe: a) $\exists$ closed immersion $Y \sqcup V \hookrightarrow X$.

b) On points: $\underset{|x| \leq 1}{\overset{\|}{X(K)}} = \underset{|x|=1}{\overset{\|}{Y(K)}} \sqcup \underset{|x|<1}{\overset{\|}{V(K)}}$,

c) Fact: $Y \sqcup V \not\cong X$!!

$Y \cup V$ is not an admissible cover of X (different G-topologies).

ADIC FIX: There are "missing points" in this theory:

"there is a point between $|x|<1$ and $|x|=1$".

$\hookrightarrow$ Gauss point.