Overconvergent modular symbols and the eigencurve

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These are notes from a talk I gave at the Seminari de Teoria de Nombres in Barcelona, January 2017, as part of a study group on Hida families. We introduce modular symbols and overconvergent modular symbols, and use them to construct the eigencurve, following work of Stevens. In particular, we define overconvergent modular symbols on suitable affinoid opens in weight space and show that any cuspidal classical eigenform can be deformed into such a family. We then define the local pieces of the eigencurve to be the spectrum of the corresponding Hecke algebra. In this way, we construct a rigid curve whose points correspond to systems of Hecke eigenvalues in the spaces of overconvergent modular symbols of varying weights. This account is essentially detail-free, and rather is an attempt to paint a broad overview of the theory.

Introduction

It is convenient to begin by immediately fixing notation. For the remainder of these notes, we take p to be a rational prime, $\Gamma = \Gamma_0(N)$with $p|N$, and $f \in S_k(\Gamma)$ a cuspidal eigenform with $U_p f = a_p f$.

In previous talks, we saw an introduction to the theory of Hida families, and in particular, showed that ordinary modular forms exist in p-adic families. Recall that an eigenform is *ordinary* if $v_p(a_p)$ is zero. The goal of these notes is to give a sketch of a generalisation of this theory. In particular, we have two, quite parallel, goals:

(1) Find familes of forms when $0 \leq v_p(a_p) < \infty$, the so-called *finite slope* case, and

(2) Give an introduction to a geometric framework for studying p-adic families, that is, the theory of *eigenvarieties.*

This more geometric framework ultimately allows vast generalisations of the theory, and turns out to be a very powerful tool.

There are two main ways of pursuing these goals. The first, through the use of Coleman's overconvergent modular forms, is more geometric in nature and predated the second, Stevens' more algebraic approach using overconvergent modular symbols. We concentrate on the latter here.

Disclaimer: as in the corresponding talk, these notes will be very heavy on sketchy arguments, hand-waving and general hand-waving, and (extremely) light on detail. They have also not been checked very carefully and are almost certain to contain mistakes; they are intended only as a snapshot of the theory. Joël Bellaïche has written some excellent and detailed notes in [Bel10], which are also very accessible, and which the author whole-heartedly recommends.

1. Weight space

We begin with a very gentle introduction to a more geometric interpretation of p-adic families. The general idea is to put the notion of 'p-adic variation of weights' onto a more solid geometric footing. In particular, let's suppose there is some 'p-adic topological space' of weights $\mathcal{W}$; it then makes sense to consider a p-adic family of 'things' to be a collection of objects $(f_\kappa)_{\kappa\in\mathcal{W}}$ such that 'f_κ varies continuously in κ.' To make this more precise, we exploit a fairly standard trick; for the correct notion of p-adic space, we should be able to consider the ring

$$R = \mathcal{O}(\mathcal{W})$$

of regular functions on $\mathcal{W}$, and note that for each element $\kappa \in \mathcal{W}$, we have an evalution map $\mathrm{ev}_\kappa : R \longrightarrow L$ (for some suitable field L of coefficients). The idea is then to define a 'big object' F over this ring R, as then the collection $(\mathrm{ev}_\kappa(F))_{\kappa\in\mathcal{W}}$ will be a p-adic family, in the vague sense given above.

In practice, we typically consider open subsets $W \subset \mathcal{W}$, and look for families over this smaller space, which is often a more tractable problem.

Example: In fact, we've already seen this phenomenon earlier in the study group. In Francesc's talk, we saw that ordinary eigenforms are the specialisations of Λ-adic forms. Here, the correct notion of weight space is

$$\mathcal{W}(\mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times),$$

which contains the integers (the 'classical weights') via the maps $z \mapsto z^k$ (and, indeed, all pairs (k,χ), where χ is a Dirichlet character of p-power conductor, via the maps $z \mapsto \chi(z)z^k$). There is a disc $W \subset \mathcal{W}(\mathbb{Z}_p)$ such that $\mathcal{O}(W) = \mathbb{Z}_p[[X]] = \Lambda$, and the evaluation map at a classical weight (k,χ) is just the map $X \mapsto \chi(u)u^k - 1$, where $u = 1+p$. In this way we see a Λ-adic form as a 'big object' parametrising its specialisations at classical weights.

Remark: The correct notion of 'p-adic topological space' for the purposes of this theory is given by *rigid geometry*, a p-adic analogue of scheme theory. It turns out that there is a rigid space $\mathcal{W}$ which, for any $\mathbb{Z}_p$-algebra L, has L-points given by

$$\mathcal{W}(L) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, L^\times).$$

To motivate this, note that we can consider a classical weight (k,χ) as a Hecke character $\psi = \chi|\cdot|^k : \mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$ in a natural way. A p-adic weight should then be a character

$$\psi : \mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{C}_p^\times.$$

Strong approximation says that $\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \cdot \mathbb{R}_{>0} \cdot \prod_\ell Z_\ell^\times$, and we see that:

(i) as $\mathbb{R}_>0$ is connected and $\mathbb{C}_p^\times$ is totally disconnected, we must have $\psi(\mathbb{R}_{>0}) = 1$, and

(ii) the incompatibility of the p-adic and ℓ-adic topologies for $\ell \neq p$ force the restriction of ψ to $\prod_{\ell\neq p}\mathbb{Z}_\ell^\times$ to factor through some finite quotient.

Accordingly, any such weight is given by its values on $\mathbb{Z}_p^\times \times (\mathbb{Z}/M)^\times$, for some integer M. We have simply restricted to the case where $M = 1$ (the so-called case of 'tame conductor 1'.

2. Modular symbols

Our aim is to give p-adic families of modular symbols. We given a brief recap of the theory. Define

$$\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})),$$

which we see as the space of paths between cusps, and note that this has a left action of Γ induced by the action $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\cdot x = (ax+b)/(cx+d)$ on $\mathbb{P}^1(\mathbb{Q})$. For a right Γ-module V, we say a homomorphism

$$\phi : \Delta_0 \longrightarrow V$$

is Γ-*invariant* if

$$\phi(\gamma D)|\gamma = \phi(D)$$

for all $\gamma \in \Gamma$ and $D \in \Delta_0$. We call the space of such functions the space of *V-valued modular symbols of level* Γ and denote this space by

$$\mathrm{Symb}_\Gamma(V) := \mathrm{Hom}_\Gamma(\Delta_0, V).$$

This definition is beautifully simple, but turns out to be incredibly powerful. In particular, for a ring R define

$$V_k(R) := \{\text{polynomials of degree at most } k \text{ over } R\},$$

which has an action of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(z) = (a+cz)^k P\left(\frac{b+dz}{a+cz}\right).$$

The dual space $V_k(R)^*$ hence inherits the dual action (on the right). To $f \in S_k(\Gamma)$, we associate

$$\begin{aligned} \phi_f : \Delta_0 &\longrightarrow V_k(\mathbb{C})^* \\ \{r\} - \{s\} &\longmapsto \left[P \longmapsto \int_r^s f(z)P(z)dz.\right] \end{aligned}$$

Then $\phi_f \in \mathrm{Symb}_\Gamma(V_k(\mathbb{C})^*)$, and in fact:

Theorem 2.1 (Eichler-Shimura)**.** *There is a Hecke equivariant isomorphism*

$$\mathrm{Symb}_\Gamma(V_k(\mathbb{C})^*) \cong S_{k+2}(\Gamma) \oplus M_{k+2}(\Gamma),$$

for the natural Hecke action on the symbol space.

Remark: It is worth elaborating on what we've actually achieved here. A modular form is an inherently *analytic* object; in a first course, they are usually introduced as holomorphic functions on the upper half-plane. Their study shows that the spaces of modular forms have an incredibly rich algebraic structure via the Hecke operators. A modular symbol, however, is a purely *algebraic* object, with a very simple definition. In passing from a modular form to its associated modular symbol, we've thrown away all of the analytic information. The Eichler-Shimura isomorphism tells us that we've not thrown away too much of the data, however; in particular, we've retained all of the Hecke information, and all of the rich algebraic structure. In this way, one could see the space of modular symbols to be the 'algebraic skeleton' of the space of modular forms. Because the definition is so simple, modular symbols often turn out to be easier to work with, nicer to compute with and – for our purposes – more friendly to vary p-adically.

3. Overconvergent modular symbols

In our attempt to vary these spaces of modular symbols in p-adic families, we encounter two immediate problems.

Problem 1: Coefficients in $\mathbb{C}$.

Our spaces of modular symbols are complex vector spaces, and it's essentially meaningless to talk about p-adically varying arbitrary complex spaces.

Solution: There is an involution ι on $\mathrm{Symb}_\Gamma(V_k(\mathbb{C}))$, given by the action of $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$, and we see that the space of modular symbols breaks up into plus and minus eigenspaces

$$\mathrm{Symb}_\Gamma(V_k(\mathbb{C})) \cong \mathrm{Symb}_\Gamma^+(V_k(\mathbb{C})) \oplus \mathrm{Symb}_\Gamma^-(V_k(\mathbb{C})).$$

We accordingly get a decomposition $\phi_f = \phi_f^+ + \phi_f^-$. Then:

Theorem 3.1 (Shimura)**.** *There exist* complex periods $\Omega_f^\pm \in \mathbb{C}^\times$ *such that*

$$\phi_f^\pm/\Omega_f^\pm \in \mathrm{Symb}_\Gamma^\pm(V_k(F))$$

for some number field F.

Henceforth, we'll replace $\phi_f^\pm$ by $\phi_f^\pm/\Omega_f^\pm$ and assume the coefficients are algebraic. Both of these symbols are eigensymbols if ϕ_f is, since the Hecke operators commute with the involution.

Problem 2: Unbounded dimension.

The key result, which underpinned all of Hida's remarkable theory of ordinary families, was that the dimension of the ordinary subspace is (essentially) independent of the weight. This is patently *not* the case in general. Indeed, the dimension of the space of modular forms of fixed level grows linearly with the weight.

Solution: We are aiming for a space of 'big objects' akin to the space of Λ-adic forms, with surjective maps to the spaces of modular symbols of any weight (at least, for any weight in an open subset of the weight space). Since these spaces have unbounded dimension, we need to pass to an infinite dimensional space. This is the space of *overconvergent modular symbols.*

Definition 3.2. For a $\mathbb{Q}_p$-algebra L, define

$$\mathcal{A}(L) := \{\text{locally analytic functions } \mathbb{Z}_p \longrightarrow L\}.$$

Here a function is *locally analytic* if it can be written locally as a convergent power series.

This has an action of the semigroup

$$\Sigma_0(p) := \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_p) : p|c, p\nmid a, ad-bc \neq 0\right\}$$

given by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot f(z) = (a+cz)^k f\left(\frac{b+dz}{a+cz}\right).$$

Note that this action is enough to give actions of both the group Γ and the Hecke operators. It should look familiar: it is precisely the action we gave $V_k(L)$ earlier. Note furthermore that $V_k(L)$ is a subset of $\mathcal{A}(L)$ and is preserved by the action of $\Sigma_0(p)$.

Note that the space $\mathcal{A}(L)$ is independent of the weight k, but that the action we put on it does depend on k. We write $\mathcal{A}(L)$ for the space with this action. Ultimately, to put these spaces into p-adic families, it is thus sufficient to p-adically interpolate this action as the weight varies.

As before, we dualise to get the coefficient system we actually want:

Definition 3.3. Define

$$\mathcal{D}_k(L) := \mathrm{Hom}_{\mathrm{cts}}\mathcal{A}(L), L)$$

with dual action

$$\mu|\gamma(f) := \mu(\gamma \cdot f).$$

Definition 3.4. Define the space of *overconvergent modular symbols* of weight k and level Γ, with coefficients in L, to be the space $\mathrm{Symb}_\Gamma(\mathcal{D}_k(L))$.

By dualising the inclusion $V_k \subset \mathcal{A}_k$, we get a ($\Sigma_0(p)$-equivariant) surjection

$$\mathcal{D}_k(L) \longrightarrow V_k(L)^*,$$

and hence a ($\Sigma_0(p)$, hence Hecke-equivariant) specialisation map

$$\rho : \mathrm{Symb}_\Gamma(\mathcal{D}_k(L)) \longrightarrow \mathrm{Symb}_\Gamma(V_k(L)^*).$$

The remarkable theorem giving meaning to all of this is the following:

Theorem 3.5 (Stevens)**.** *Let $f \in S_{k+2}(\Gamma)$ be an eigenform, with associated modular symbols $\phi_f^\pm \in \mathrm{Symb}_\Gamma^\pm(V_k(L))$, for some sufficiently large $L/\mathbb{Q}_p$. Then there exists*

$$\psi_f^\pm \in \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_k(L))$$

such that

$$\rho(\psi_f^\pm) = \phi_f^\pm.$$

If $v_p(a_p) < k+1$, then $\psi_f^\pm$ is unique.

Remark: This is actually a butchering together of *two* remarkable theorems of Stevens, both found in [PS12]. The first, his *control theorem*, says that the specialisation map is an isomorphism on the *slope $< k+1$ subspaces*, whilst the second says that there is (almost![1]) a bijection between systems of Hecke eigenvalues occuring in $\mathrm{Symb}_\Gamma(\mathcal{D}_k)$ and the systems of Hecke eigenvalues occuring in the space of overconvergent modular forms of weight $k+2$. For our purposes, it's enough to note that the system of eigenvalues attached to any classical eigenform shows up in the corresponding space of overconvergent modular symbols.

4. Modular symbols in p-adic families

Throughout this section, we'll fix a nice *open affinoid* $W = \mathrm{Sp}(R) \subset \mathcal{W}$ to work over. Here:

- R will be a $\mathbb{Q}_p$-affinoid algebra; the reader who hasn't seen these before should just think of R as being some power series ring over $\mathbb{Q}_p$.
- $\mathrm{Sp}(R)$ means the max spectrum of R, together with the associated rigid structure, which we'll not specify.
- In close analogy to the theory of schemes, R is the space of rigid functions $\mathcal{O}(W)$ on W.

[1] There is, rather bemusingly, a single exception. There is a system of eigenvalues, which Stevens calls E_2^{crit}, that appears in $\mathrm{Symb}_\Gamma^-(\mathcal{D}_0(\mathbb{Q}_p))$, but not in the space of overconvergent modular forms of weight 2. This system of eigenvalues, also rather bemusingly, then doesn't appear in the eigencurve of modular symbols; it is not the specialisation of any family of modular symbols. Perhaps there's an obvious reason why these mutual bemusements should cancel out, but it is not apparent to the author!

The take-home message is that we've now got a big ring R, which is a $\mathbb{Q}_p$-algebra, so we can talk about $\mathcal{A}(R)$ and $\mathcal{D}(R)$. As we alluded earlier, the key step to varying spaces of overconvergent modular symbols in families is going to be to interpolate the weight k actions of $\Sigma_0(p)$ as p varies, and in particular, to answer:

Key question: Can we make sense of $\mathrm{Symb}_\Gamma(\mathcal{D}(R))$? That is, can we define an action of $\Sigma_0(p)$ on $\mathcal{D}(R)$ that commutes with the natural evaluation maps ev_k and the action of $\Sigma_0(p)$ on $\mathcal{D}_k(L)$ at each $k \in W$?

If the answer to this is *yes* (which, of course, it is), then an element of this space would play the role of the 'big objects' from earlier, and we'd have the notion of p-adic families of overconvergent modular symbols coming from evaluation maps $\mathrm{ev}_k : R \to L$ inducing maps

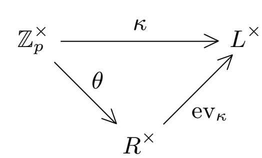
$$\mathrm{ev}_k : \mathrm{Symb}_\Gamma(\mathcal{D}(R)) \longrightarrow \mathrm{Symb}_\Gamma(\mathcal{D}_k(L)).$$

Hence answering this question in the affirmative gives the p-adic families we desire. The key observation is the following.

Definition 4.1. The inclusion $W \to \mathcal{W}$ means that $W \in \mathcal{W}(R) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, R^\times)$ is an R-valued point of weight space[2]. Define θ to be the corresponding structure homomorphism

$$\theta : \mathbb{Z}_p^\times \longrightarrow R^\times.$$

Remark: Note that any element $\kappa \in W(L) \subset \mathcal{W}(L)$ is a max ideal of R, and hence gives rise to a (quotient) map $\mathrm{ev}_\kappa : R \to L$, where L is the residue field (in this setting, we say that κ is *defined over* L). Since $\kappa \in \mathcal{W}(L)$, it also corresponds to a homomorphism $\kappa : \mathbb{Z}_p^\times \to L^\times$ by definition. The map θ is universal in the sense that the diagram



commutes, that is,

$$\kappa = \mathrm{ev}_\kappa \circ \theta.$$

In this way, we see θ as parametrising the weight characters $\mathbb{Z}_p^\times \to L^\times$ for the elements of $W(L)$.

Example: At risk of labouring the point, he most pertinent example is the following: suppose the classical weight $k \in W(L)$, and then note that when $z \in \mathbb{Z}_p$, $a \in \mathbb{Z}_p^\times$ and $c \in p\mathbb{Z}_p$, we have $a + cz \in \mathbb{Z}_p^\times$ and

$$\mathrm{ev}_k \circ \theta(a + cz) = (a + cz)^k.$$

Now we can equip the space $\mathcal{A}(R)$ (of R-valued locally analytic functions on $\mathbb{Z}_p$) with an action of $\Sigma_0(p)$ as follows; define

$$(\gamma \cdot f)(z) := \theta(a + cz) f\left(\frac{b + dz}{a + cz}\right).$$

[2] Whilst this is a standard notion, the author, who is not particularly geometrically minded, found this to be a somewhat unnatural thing to get his head around. If any readers have a similar problem, the following analogy might be comforting: given a scheme S, a k-valued point of S is a morphism $\mathrm{Spec}(k) \to S$.

Remark: This interpolates the classical weight k actions in the following sense; viewing k as an element of $W(L)$, we define $\mathrm{ev}_k(f) \in \mathcal{A}(L)$ by composing $f : \mathbb{Z}_p \to R$ with the $\mathrm{ev}_k : R \to L$, and we see that

$$\begin{aligned}\mathrm{ev}_k(\gamma \cdot f)(z) &= \mathrm{ev}_k\left(\theta(a+cz)f\left(\frac{b+dz}{a+cz}\right)\right)\\ &= (a+cz)^k \mathrm{ev}_k(f)\left(\frac{b+dz}{a+cz}\right) = [\gamma \cdot \mathrm{ev}_k(f)](z).\end{aligned}$$

By dualising, we get a natural action of $\Sigma_0(p)$ on $\mathrm{Hom}(\mathcal{A}(R), R)$.

Definition 4.2. In a technical twist, define

$$\mathcal{D}(R) := \mathcal{D}(\mathbb{Q}_p)\widehat{\otimes}_{\mathbb{Q}_p} R,$$

rather than the dual space $\mathrm{Hom}(\mathcal{A}(R), R)$. (Apologies for the somewhat abrupt departure from the previously agreed notation).

There is a natural injection $\mathcal{D}(R) \hookrightarrow \mathrm{Hom}(\mathcal{A}(R), R)$, and this space is preserved under the action of $\Sigma_0(p)$ (see [Bel12], Remark 3.1). Moreover, there are clear specialisation maps $\mathcal{D}(R) \to \mathcal{D}_k(L)$ for any classical weight $k \in W(L)$, which respect the actions of $\Sigma_0(p)$ (by the above remark). Hence:

Answer to key question: Yes!

Definition 4.3. A *p-adic family of (overconvergent) modular symbols over* $W = \mathrm{Sp}(R) \subset \mathcal{W}$ is an element of $\mathrm{Symb}_\Gamma(\mathcal{D}(R))$.

A natural question now arises; which systems of Hecke eigenvalues can be put into a family in this way?

Short answer:

Theorem 4.4 (Stevens)**.** *Let* $f \in S_{k+2}(\Gamma)$ *be a classical eigenform. Then the system of eigenvalues corresponding to* f *appears as the specialisation under* ev_k *of a family of modular symbols over some affinoid* W *in* $\mathcal{W}$ *containing* k*.*

Slightly more detailed answer:

Let's be optimistic, and hope for the following:

Hope 4.5. *Let* $\psi_f \in \mathrm{Symb}_\Gamma(\mathcal{D}_k(L))$ *be an overconvergent modular eigensymbol. There exists an affinoid* $W = \mathrm{Sp}(R)$ *of the form above and an element* $\psi_R \in \mathrm{Symb}_\Gamma(\mathcal{D}(R))$ *that specialises to* ψ_f *at weight* k*.*

Here, however, we fall short and encounter reality.[3]

[3] Recall that the system of eigenvalues E_2^{crit}, which is defying expectations by refusing to live in a family, is the exact same exceptional system that showed up for overconvergent modular symbols but not for overconvergent modular forms. Ultimately, this 'cancelling' actually works to our advantage, since it means that we don't get an extra point in the eigencurve of modular symbols.

Theorem 4.6 (Stevens)**.** *Hope 4.5 is true* unless $k = 0$ *and* ψ_f *corresponds to the system of eigenvalues* E_2^{crit}.

We'll give a sketch of the proof of this below, but first, the relevant lines proving Theorem 4.4:

Proof. Theorem 4.4. This follows immediately from the previous results/observations that:

(i) the system of eigenvalues appears as $\phi_f^{\pm}$ in the space of modular symbols with coefficients in L for $L/\mathbb{Q}_p$ sufficiently large,

(ii) it hence appears as $\psi_f^{\pm}$ in the space of overconvergent modular symbols,

(iii) and it is not the system E_2^{crit}, which is not classical.

Note that actually we can put both ψ_f^+ and ψ_f^- into p-adic families ψ_R^+ and ψ_R^-; we don't need to be picky when it comes to cusp forms! □

Proof. Theorem 4.6; sketch. We prove this theorem using cohomology. In particular, for (almost) any right Γ-module V we have a functorial isomorphism

$$\mathrm{H}^1_{\mathrm{c}}\big(Y_\Gamma, \widetilde{V}\big) \cong \mathrm{Symb}_\Gamma(V)$$

(see [AS86]). In addition, at any fixed weight this specialisation map is surjective, and we have the exact sequence

$$0 \to \mathcal{D}(R) \xrightarrow{\times u_k} \mathcal{D}(R) \to \mathcal{D}_k(L) \to 0,$$

where $L = R/(u_k)$ for the maximal ideal $(u_k) \subset R$ corresponding to k. Accordingly we can study the specialisation map through the long exact sequence of cohomology attached to this short exact sequence. This gives rise to an exact sequence

$$\mathrm{H}^1_{\mathrm{c}}\big(Y_\Gamma, \widetilde{\mathcal{D}(R)}\big) \xrightarrow{u_k} \mathrm{H}^1_{\mathrm{c}}\big(Y_\Gamma, \widetilde{\mathcal{D}(R)}\big) \longrightarrow \mathrm{H}^1_{\mathrm{c}}\big(Y_\Gamma, \widetilde{\mathcal{D}_k(L)}\big) \longrightarrow \mathrm{H}^2_{\mathrm{c}}\big(Y_\Gamma, \widetilde{\mathcal{D}(R)}\big)[u_k].$$

Via Poincaré duality, we have

$$\mathrm{H}^2_{\mathrm{c}}\big(Y_\Gamma, \widetilde{\mathcal{D}(R)}\big) \cong \mathrm{H}_0(\Gamma, \mathcal{D}(R)),$$

where we've identified the topological homology of Y_Γ with the group homology of Γ in the usual way. This is the space of Γ-coinvariants of $\mathcal{D}(R)$, and in particular is nothing but $\mathcal{D}(R)/I_\Gamma\mathcal{D}(R)$, where I is the augmentation ideal of the group ring $\mathbb{Z}[\Gamma]$. We now use:

Lemma 4.7. *There is an isomorphism*

$$\mathcal{D}(R)/I_\Gamma\mathcal{D}(R) = \begin{cases} \mathbb{Q}_p & : 0 \in W(\mathbb{Q}_p), \\ 0 & : \textit{otherwise.} \end{cases}$$

Proof. See [Bel10], Lemma IV.1.16. □

As an immediate corollary, we see that in the above set-up, the specialisation map $\mathrm{Symb}_\Gamma(\mathcal{D}(R)) \to \mathrm{Symb}_\Gamma(\mathcal{D}_k(L))$ is surjective if $k \neq 0$. If $k = 0$, then the cokernel is one-dimensional, and can be shown to be spanned by E_2^{crit}. This completes the proof. □

5. The eigencurve

We now know that classical systems of eigenvalues live in p-adic families of modular symbols, in the sense that there exists some 'big modular symbol' $\psi_R \in \mathrm{Symb}_\Gamma(\mathcal{D}(R))$ specialising to an overconvergent modular symbol with the same system of eigenvalues. What we have not said, though, is anything about the behaviour of ψ_R under the Hecke operators. Obviously it's very desirable that this should be an eigensymbol in the natural sense, so that ψ_R actually parametrises a p-adic family of *eigensymbols*. One can prove that this big symbol *is* an eigensymbol, and one way of doing, due to Bellaïche so is to invoke the theory of *eigenvarieties*.

This, the last section of these notes, uses what we've done so far to give an introduction to the eigencurve constructed using modular symbols. The reader is warned that, even by the sketchy standards we've been adhering to thus far, for the remainder we'll dispense with the technical details altogether. The result, then, should be viewed simply as a sketch of what the eigencurve should be. For the details, see [Buz07] (for the general eigenvariety machine) or [Bel10] (for the eigencurve of modular symbols).

Consider the following motivation. Let's pretend we're equipped with p-adic paper so that we can draw p-adic graphs, and use it to plot systems of Hecke eigenvalues appearing in the space of overconvergent modular symbols. The x-axis will be the weight space $\mathcal{W}$, and the y-axis will be the *slope* of the system, that is, the (non-negative) real number $v_p(a_p)$. The hope is that, as we plot all of these points, the result is that we obtain something geometrically 'nice' (I intended to draw a picture at this point, but failed to get round to it).

A point of this 'eigencurve' lies above some weight $\kappa \in \mathcal{W}$, and – by definition – corresponds to some system of Hecke eigenvalues appearing in $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa)$, that is, an eigensymbol in this space. Coversely, any such eigensymbol gives rise to a point 'by construction.' This motivates:

> **Aims:** *Find a p-adic (rigid analytic) curve $\mathcal{C}$, together with a 'weight map' $\kappa : \mathcal{C} \longrightarrow \mathcal{W}$, such that there is a bijection*
>
> $$\{\textit{Points } x \in \mathcal{C}(L) \textit{ with } \kappa(x) = k\} \longleftrightarrow \{\textit{Systems of Hecke eigenvalues occurring in } \mathrm{Symb}_\Gamma(\mathcal{D}_k(L))\}. \tag{1}$$

In the remainder of these notes, we sketch how to make this aim a reality. We'll use Hecke algebras.

Definition 5.1. Define the (abstract) Hecke algebra to be the $\mathbb{Z}$-algebra generated by the Hecke operators, that is,

$$\mathcal{H} := \mathbb{Z}[\{T_\ell : \ell \nmid N\}, U_p, \{\langle a \rangle : a | N\}].$$

This ring acts on basically everything in sight, and in particular, it acts on all of the spaces mentioned previously. In fact, we have more.

Crucial fact: The U_p operator acts compactly on all the spaces we've considered. *Without this, the whole theory would fall apart*[4].

[4]It's worth pointing out that it is this which forces us to consider *overconvergent* modular forms in the analytic setting. If we defined p-adic modular forms in perhaps the most natural way, as the p-adic completion of the (integral) space of modular forms, then what we end up with is *far too big*, and in particular the U_p operator fails to act compactly. By demanding that these forms 'overconverge', the problem goes away.

In short, this means that the U_p operator comes with a nice spectral theory, and in particular a discrete spectrum of eigenvalues. A key consequence is the following:

Corollary 5.2. *Let $v \in \mathbb{R}_{>0}$ and fix some classical weight k.*

(i) There exists a slope decomposition for U_p*, that is, a direct summand*

$$\mathrm{Symb}_\Gamma(\mathcal{D}_k(L))^{\leq v} \subset \mathrm{Symb}_\Gamma(\mathcal{D}_k(L))$$

that is a finite dimensional vector space over L.

(ii) In fact, there also exists $W = \mathrm{Sp}(R) \subset \mathcal{W}$, containing k, such that there exists a slope decomposition

$$\mathrm{Symb}_\Gamma(\mathcal{D}(R))^{\leq v} \subset \mathrm{Symb}_\Gamma(\mathcal{D}(R)),$$

which is a finite flat R-module.

Colloquially, these are the subspaces where 'U_p acts with $v_p(a_p) \leq v$'.

So we have nice finite structures. The Hecke operators also preserve slope decompositions, so we have actions on these spaces too.

Definition 5.3. Define

$$\mathbb{T}^{\pm}_{W,v} := \text{image of } \mathcal{H} \text{ in } \mathrm{End}(\mathrm{Symb}^{\pm}_\Gamma(\mathcal{D}(R))^{\leq v}.$$

Similarly, define

$$\mathbb{T}^{\pm}_{k,v} := \text{image of } \mathcal{H} \text{ in } \mathrm{End}(\mathrm{Symb}^{\pm}_\Gamma(\mathcal{D}_k(L))^{\leq v}.$$

These are finite flat modules over R and L respectively.

Now for the key definition:

Definition 5.4. Define the *local piece of the eigencurve* (over W of slope $\leq v$) to be

$$\mathcal{C}^{\pm}_{W,v} := \mathrm{Sp}(\mathbb{T}^{\pm}_{W,v}).$$

We have a natural weight map $\kappa : \mathcal{C}^{\pm}_{W,v} \rightarrow W$ induced by the structure map $R \rightarrow \mathbb{T}^{\pm}_{W,v}$ by functoriality.

So why does this do what we want? Let $x \in \mathcal{C}^{\pm}_{W,v}(L)$ be an L-point, that is, a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{\pm}_{W,v}$ with residue field L. The quotient map is an algebra homomorphism

$$\widetilde{\phi} : \mathbb{T}^{\pm}_{W,v} \longrightarrow \mathbb{T}^{\pm}_{W,v}/\mathfrak{m} = L.$$

Suppose that $\kappa(x) = k \in W(L)$, which says precisely that this factors through the localisation $\mathbb{T}^{\pm}_{W,v} \otimes_{k,L} L \subset \mathbb{T}^{\pm}_{k,v}$ at the prime of R corresponding to k, that is, that there exists some algebra homomorphism ϕ such that $\widetilde{\phi}$ is the composition

$$\mathbb{T}^{\pm}_{W,v} \longrightarrow \mathbb{T}^{\pm}_{k,v} \xrightarrow{\ \phi\ } L.$$

But such a homomorphism, by definition, corresponds to a system of eigenvalues occuring in $\mathrm{Symb}^{\pm}_\Gamma(\mathcal{D}_k(L))$. So we get such a system.

What about the converse? Well, let ψ be an eigensymbol in $\mathrm{Symb}^{\pm}_\Gamma(\mathcal{D}_k(L))$, giving rise to an algebra homomorphism $\mathbb{T}^{\pm}_{k,v} \rightarrow L$. Assume ψ is not E_2^{crit}, so that we know that ψ is in the image of specialisation from $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}(R))$ (for some R). Note also that $\mathbb{T}_{W,v}^\pm \otimes_{k,L} L$ is simply the image of $\mathbb{T}_{W,v}^\pm$ under the weight k evaluation map. Accordingly, we have

$$\mathbb{T}_{W,v}^\pm \xrightarrow{\mathrm{ev}_k} \mathbb{T}_{W,v}^\pm \otimes_{k,L} L \subset \mathbb{T}_{k,v}^\pm \xrightarrow{\phi} L,$$

and the condition that ψ is in the image of specialisation shows that the composition is non-zero. Hence we obtain a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{W,v}^\pm$, and hence a point of $\mathcal{C}_{W,v}^\pm(L)$.

Hence the local piece of the eigencurve has the properties we wanted, away from E_2^{crit}! We then conclude by stating a gluing result:

Theorem 5.5 (Stevens (this case), Buzzard (general machine))**.** *The local pieces $\mathcal{C}_{W,v}^\pm$ can be glued together into rigid curves $\mathcal{C}^\pm$, and the weight maps glued into a weight map $\kappa : \mathcal{C}^\pm \to \mathcal{W}$, with the properties of (1) above (though, of course, excluding the system of eigenvalues E_2^{crit}).*

Remarks: (i) It would be remiss to talk about the eigencurve in any capacity and not mention the original results of Coleman and Mazur, who gave the original construction using overconvergent modular forms. They construct a cuspidal eigencurve $\mathcal{C}^0$ living inside the eigencurve $\mathcal{C}$ itself. Results of Chenevier show that in fact

$$\mathcal{C}^0 \subset \mathcal{C}^\pm \subset \mathcal{C}, \quad \mathcal{C}^+ \cup \mathcal{C}^- = \mathcal{C}.$$

(ii) As promised, I've failed to mention any of the details. In particular, I've failed to mention some of the modules we actually need to carry out Buzzard's machine – in particular, the Banach module of distributions of order r – and the compatibility of symbol spaces over affinoids with changing the affinoid (Buzzard's theory of *links*).

(iii) The 'eigenvariety' version of the theory generalises hugely to other settings!

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