p-Adic L-Functions for GL(3)

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§1: Set-up/context

Let – π = regular algebraic cuspidal automorphic rep'n of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$,
$\longrightarrow$ attached L-function $L(\pi, s)$ centred at $s = 1/2$,
– π_p unramified + ordinary.

Fundamental question in the Iwasawa theory of π:

Q: When can we construct a p-adic L-function for π? ... A: in frustratingly few cases.

$n = 1, 2$: ✓ (60s, 70s)

$n > 2$: all known examples require π transfer from a "smaller" group

e.g. symmetric squares, transfer from $\mathrm{GSp}_4/\mathrm{GSp}_{2n+1}$, Asai transfer, Rankin–Selberg transfer, ...

Today: construction for $n = 3$. (... with no transfer assumption).

§2: The Coates–Perrin-Riou conjecture for GL(3)

Take $n = 3$. Assume π has trivial central character.

(I): Algebraicity WLOG π has weight $(a, 0, -a)$, some $a \in \mathbb{Z}_{\geq 0}$. Let

exchanged by Func. eqn!

$$\mathrm{Crit}^-(\pi) = \{(\chi, j) : \chi \text{ Dir. char}, -a \leq j \leq 0, \quad \chi(-1)(-1)^j = 1\}$$
$$\mathrm{Crit}^+(\pi) = \{(\chi, j) : \text{''} \quad \text{''}, \quad 1 \leq j \leq a+1, \quad \chi(-1)(-1)^j = -1\}$$

Conjecture A: (Deligne, '79) $\exists$ complex periods $\Omega_\pi^\pm \in \mathbb{C}^\times$ such that $\forall (\chi, j) \in \mathrm{Crit}^\pm(\pi)$,

$$\underset{\overline{\mathbb{Q}}^\times \cdot (2\pi i)^{\mathbb{Z}}}{\underset{\cap}{e_\infty(\pi, j)}} \frac{L(\pi \times \chi, j)}{\Omega_\pi^\pm} \in \overline{\mathbb{Q}}.$$

Remark: By the Func. eqn., STP Conj A. for $\mathrm{Crit}^-(\pi)$.

Theorem: (Mahnkopf '98; Kasten–Schmidt '13) $\exists\ \Omega_\pi^- \in \mathbb{C}^\times$ and (inexplicit) $\tilde{e}_\infty(\pi, j) \in \mathbb{C}^\times$ s.t.

$$\tilde{e}_\infty(\pi, j) \frac{L(\pi \times \chi, j)}{\Omega_\pi^-} \in \overline{\mathbb{Q}} \qquad \forall (\chi, j) \in \mathrm{Crit}^-(\pi).$$

(II) Interpolation

Let

$$\mathrm{Crit}_p^-(\pi) = \left\{ (\chi, j) \in \mathrm{Crit}^-(\pi) : \mathrm{cond}(\chi) \mid p^\infty \right\}.$$

Let $\mathcal{O} :=$ p-adic integer ring containing Hecke eigenvalues of π.

<u>Conjecture B</u>: (Coates – Perrin-Riou, '89) There exists

$$L_p^-(\pi) \in \mathcal{O}[\![\mathbb{Z}_p^\times]\!] \quad \leftarrow \mathcal{O}\text{-valued measures on } \mathbb{Z}_p^\times$$

s.t. $\forall (\chi, j) \in \mathrm{Crit}_p^-(\pi)$, have

$$L_p^-(\pi)\big(\underbrace{\chi(x)x^{-j}}_{\text{function on } \mathbb{Z}_p^\times}\big) = \underbrace{e_p(\pi, \chi, j)}_{\text{modified Euler factor at } p} \cdot e_\infty(\pi, j) \cdot \frac{L(\pi \times \chi, j)}{\Omega_\pi^-}.$$

<u>Theorem</u>: (Loeffler – W.) Conjecture B holds.

<u>Remark</u>: We assume only that π_p is P-ordinary, $P = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} \subset GL_3$ (Panchishkin's conjecture).

+ allow π_p ramified.

<u>Corollary</u>: Conjecture A holds.

Rest of talk: wt $(0,0,0)$ $\Longleftrightarrow$ interpolate $\left\{ L(\pi \times \chi, 0) : \chi \text{ even}, \ \mathrm{cond}(\chi) \mid p^\infty \right\}$.

§3: Mahnkopf's work

Recall: Mahnkopf proved Conj A for $n=3$

... via $GL_3 \times GL_2$ Rankin – Selberg convolution

<u>Def'n</u>: Let E_χ = GL_2 Eisenstein series of wt 2, char. χ

$\in I(|\cdot|^{-1/2}\chi, |\cdot|^{1/2})$ (explicit element built from Schwartz functions).

Fact: $L(\pi \times E_\chi, \tfrac{1}{2}) = L(\pi\times\chi, 0)\cdot L(\pi, 1).$

Cohomological analogue.

$$\begin{tikzcd}
GL_2 \arrow[r, squiggly] & Y^{GL_2} = _\backslash GL_2(\mathbb{A})/_\, Z_{GL_2} & (2\text{-dim}) \\
H := GL_2\times GL_1 \arrow[u, "pr"] \arrow[d, "\iota:\ (g,z)\mapsto \begin{pmatrix} g & \\ & z\end{pmatrix}"'] \arrow[r, squiggly] & \tilde{Y}^H = _\backslash H(\mathbb{A})/_\, \iota^{-1}(Z_{GL_3}) & (3\text{-dim}) \\
GL_3 \arrow[r, squiggly] & Y^{GL_3} = _\backslash GL_3(\mathbb{A})/_\, Z_{GL_3} & (5\text{-dim})
\end{tikzcd}$$

(4)

Note:

$$\begin{tikzcd}
\pi \arrow[ddd, squiggly, "\text{(2) } \exists \text{ class}"'] & \times & E_\chi \arrow[d, squiggly, "\text{(1) } \exists \text{ class, ``modular symbol''}"] \\
 & & \phi_\chi \in H^1(Y^{GL_2}, \mathbb{C}) \arrow[d, "pr^*"] \\
 & & H^1(\tilde{Y}^H, \mathbb{C}) \arrow[d, "\iota_*"] \quad (4) \\
\phi_\pi \in H^2_c(Y^{GL_3}, \mathbb{C}) & \times & H^3(Y^{GL_3}, \mathbb{C}) \longrightarrow \mathbb{C}
\end{tikzcd}$$

(3) $\exists$ pairing $\langle\ ,\ \rangle : H^2_c(Y^{GL_3}, \mathbb{C}) \times H^3(Y^{GL_3}, \mathbb{C}) \longrightarrow \mathbb{C}$

Theorem: (Mahnkopf)

$$\langle \phi_\pi, \iota_* pr^* \phi_\chi \rangle = \overset{\text{explicit}}{(*)}\, L(\pi\times E_\chi, \tfrac{1}{2})$$
$$= [(*)\, L(\pi, 1)] \cdot \underbrace{L(\pi\times\chi, 0)}_{\text{target } L\text{-value!}}$$

For Conjecture A: put algebraic structure on the cohomology groups,

$$\Omega_\pi^- = \frac{[\text{whittaker period}]}{L(\pi,1)} .$$

§4: Interpolation

Aim: p-adically interpolate $\iota_* pr^* \phi_\chi$ as χ varies.

Picture:

$$\begin{tikzcd}
(\phi_\pi, \quad \xi \quad) \arrow[rrr, mapsto] & & & \text{Aim: } L_p^-(\pi) \; ②' \arrow[ddd, mapsto] \\
H^2(Y^G, \mathcal{O}) \times \left(H^3(Y^G, \mathbb{Z}_p) \otimes \mathbb{Z}_p[[\mathbb{Z}_p^\times]]\right) \arrow[rr, "{\langle , \rangle}"] \arrow[d, "\mathrm{id}"'] \arrow[d, shift left=20ex, "③\ \mathrm{id} \otimes \chi"] & & \mathcal{O}[[\mathbb{Z}_p^\times]] \arrow[d, "② \quad \mu \mapsto \mu(\chi)"] \arrow[ur, mapsto] & \\
H^2_c(Y^G, \mathcal{O}) \times H^3(Y^G, \overline{\mathcal{O}}) \arrow[rr, "{\langle , \rangle}", "①"'] & & \overline{\mathcal{O}} & \\
(\phi_\pi \ , \quad \iota_* pr^* \phi_\chi) \; ①' \arrow[rrr, mapsto] & & & (*) \cdot L(\pi \times \chi, 0)
\end{tikzcd}$$

Theorem: (Loeffler–W). For $r \geq 1$, $\exists$ classes $\xi_r \in H^3(Y^G, \mathbb{Z}) \otimes \mathbb{Z}[(\mathbb{Z}/p^r)^\times]$ s.t.:

① $\forall$ even χ of cond $\mid p^r$, have

$$\begin{tikzcd}
H^2_c(Y^G, \mathcal{O}) \times \left(H^3(Y^G, \mathbb{Z}_p) \otimes \mathbb{Z}_p[(\mathbb{Z}/p^r)^\times]\right) \arrow[r, "{\langle , \rangle}"] & \mathcal{O}[(\mathbb{Z}/p^r)^\times] \arrow[d, "\chi"] \\
(\phi_\pi \ , \ \xi_r) \arrow[r, mapsto] & \alpha^r e_p \cdot e_\infty \cdot \frac{L(\pi \times \chi, 0)}{\Omega_\pi^-} \in \overline{\mathcal{O}},
\end{tikzcd}$$

where $U_p \phi_\pi = \alpha \phi_\pi$, $v_p(\alpha) = 0$.

② $\mathrm{Norm}_r^{r+1} \, \xi_{r+1} = U_p \cdot \xi_r$. $\rightsquigarrow$ "Betti Euler system".

Corollary: $L_p^-(\pi) := \left(\alpha^{-r} \langle \phi_\pi, \xi_r \rangle\right)_{r \geq 1} \in \varprojlim_r \mathcal{O}[(\mathbb{Z}/p^r)^\times] \cong \mathcal{O}[[\mathbb{Z}_p^\times]]$

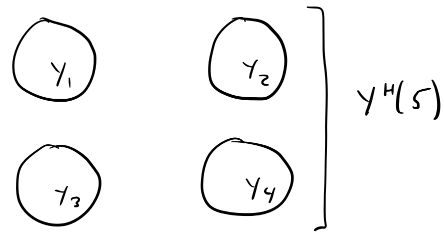
is the p-adic L-function predicted by Coates–Perrin-Riou.

§5: Constructing $\mathfrak{Z}_r$

... where does the group algebra come from?

<u>Fact:</u> We have

$$\tilde{Y}^H(p^r) = \coprod_{a \in (\mathbb{Z}/p^r)^\times} \tilde{Y}^H_a(p^r).$$



$$\mathrm{Eis}^{mot}_{p^n} \in H^1_{mot}(Y^{GL_2}(p^r), \mathbb{Z})$$

$$\big\downarrow \text{ Betti realization + base-change}$$

$$H^1(Y^{GL_2}(p^r), \mathbb{Z}_p)$$

$$\big\downarrow pr^*$$

$$H^1(\tilde{Y}^H(p^r), \mathbb{Z}_p) = \bigoplus_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} H^1(\tilde{Y}^H_a(p^r), \mathbb{Z}_p)$$

$$\big\downarrow \quad \phi = (\phi_a) \mapsto \sum \phi_a \otimes [a]$$

$$H^1(Y^H(p^r), \mathbb{Z}_p) \otimes \mathbb{Z}_p[(\mathbb{Z}/p^r)^\times]$$

$$\big\downarrow \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}_* \circ \iota_*$$

"variation of ι"

$$H^3(Y^{GL_3}(p^r), \mathbb{Z}_p) \otimes \mathbb{Z}_p[(\mathbb{Z}/p^r)^\times]$$

$$\big\downarrow \begin{pmatrix} p^r & & \\ & 1 & \\ & & 1 \end{pmatrix}_*$$

leads to U_p operator

$$\mathfrak{Z}_r \in H^3(Y^{GL_3}, \mathbb{Z}_p) \otimes \mathbb{Z}_p[(\mathbb{Z}/p^r)^\times]$$

(Key: integral + nm compat).

Input: 1) use Beilinson's motivic Eisenstein classes $\rightsquigarrow$ systematic integrality + norm-compat.

2) To preserve norm-compat: twist embedding ι by $\begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$

$\rightsquigarrow$ (2) holds by spherical varieties formalism.

3) Explicit check:

$$\langle \phi_\pi, \; z_r(\chi) \rangle \sim \text{ global R-S integral for } \pi \times E_\chi .$$

4) Computation of local zeta at p: gives C-PR factor. $(\square)$