NON-VANISHING OF p-REFINED FRIEDBERG–JACQUET INTEGRALS

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(WITH DANIEL BARRERA AND ANDREW GRAHAM)

Abstract. Let π be an automorphic representation of $G = \mathrm{GL}(2n)$, and let $H = \mathrm{GL}(n) \times \mathrm{GL}(n)$ diagonally embedded. By work of Friedberg–Jacquet and Asgari–Shahidi, period integrals for π over H are non-vanishing only when π is a global functorial transfer from $\mathrm{GSpin}(2n+1)$.

I will discuss joint work with Daniel Barrera and Andrew Graham, where we predict a local 'p-refined' analogue of this result, arising from the study of p-adic L-functions and p-adic families. When studying p-adic variation, one is led naturally to 'P-twisted' local Friedberg–Jacquet integrals at p. I will describe a conjectural criterion for non-vanishing of this local integral (and hence of p-adic L-functions constructed via Shalika methods) in terms of being a local transfer from $\mathrm{GSpin}(2n+1)$, and describe some results towards this conjecture. The proofs of these results go through the study of the geometry of the $\mathrm{GL}(2n)$-eigenvariety at points attached to π.

These are notes from a talk given at RIMS[1] in January 2025 on the paper [BGW25]:

> D. Barrera, A. Graham, and C. Williams: *On p-refined Friedberg–Jacquet integrals and the classical symplectic locus in the* GL*(2n) eigenvariety.*
>
> Research in Number Theory, 11 (2025), no.51, pp. 1-57.

The primary focus of this paper was the study of the geometry of the GL(2n) eigenvariety at symplectic points. It predicted the dimensions of (or in some case the non-existence of) classical families through such points, inspired by analogous conjectures of Ash–Pollack–Stevens [APS08] for GL(3) and Calegari–Mazur [CM09] for Bianchi modular forms. However, this note will have the same focus as my RIMS talk, namely on our application to Friedberg–Jacquet integrals from Part III of the paper.

I will at times be loose with details, or leave tangential objects undefined, so as not to distract from the narrative/flow of the note. Particularly notable is that I specialise to $s = 1/2$ everywhere without further comment, suppressing details around convergence. For a completely precise account the reader is referred to the above paper and the references therein.

1. Period integrals

By way of motivation, I will first introduce the period integrals of interest in wide generality, and highlight some of the interesting arithmetic connections they possess.

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[1]I am grateful to Kenichi Namikawa, both for giving the opportunity to speak at RIMS and for generously funding my trip to Kyoto.

Let G be a reductive group, and $H \leq G$ a reductive subgroup. Let π be a cuspidal automorphic represenation of $G(\mathbb{A})$. The primary object of interest in this note is the *period integral*

$$P_H(\varphi,\psi) := \int_{[H]} \varphi(h)\psi(h)dh,$$

where:

- $\varphi \in \pi$ is some choice of cusp form,
- $[H] = (Z_G \cap H)(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})$ with Z_G the centre of G,
- and ψ is a character of $[H]$.

If ψ is trivial, we omit it, and simply write $P_H(\varphi)$.

These integrals have been studied in many contexts, forming the basis for varying and beautiful results. A slogan might be:

> *"If there exist φ and ψ such that $P_H(\varphi,\psi) \neq 0$, then we can deduce important arithmetic consequences."*

If $P_H(\varphi) \neq 0$ for some $\varphi \in \pi$, then we usually say π is *H-distinguished*, and a cleaner slogan is that *"being H-distinguished often characterises interesting arithmetic properties"*. However for the purposes of this note I want to emphasise the non-vanishing, and the character ψ is essential, so I'll not use this terminology.

Let us give some examples of this phenomenon.

Examples 1: (1) (Gan–Gross–Prasad). Let $G = U(n) \times U(n+1)$ and $H = U(n)$, diagonally embedded. Then the GGP conjecture [GGPW12], now a theorem by [BPCZ22, BPLZZ21], says that

$$\exists\varphi \in \pi \text{ such that } P_H(\varphi) \neq 0 \iff \Big[L(\pi,\tfrac{1}{2}) \neq 0 \quad \text{and} \quad \mathrm{Hom}_H(\pi,\mathbb{C}) \neq 0.\Big]$$

(2) (Asai, Flicker–Rallis) Let F be an imaginary quadratic field, $G = \mathrm{GL}_{2,F}$, and $H = \mathrm{GL}_{2,\mathbb{Q}}$. In this case π corresponds to a Bianchi modular form. Then

$$\exists\varphi \in \pi \text{ such that } P_H(\varphi) \neq 0 \iff \pi \text{ is base-change from } H.$$

This goes through relating $P_H(\varphi)$ to a residue of the Asai L-function of π – studied in [Asa77] – which has a pole if and only if π is base-change.

(3) (Friedberg–Jacquet) Let $G = \mathrm{GL}_{2n}$ and $H = \mathrm{GL}_n \times \mathrm{GL}_n$, diagonally embedded. Let χ and η be Dirichlet characters, and let

$$\psi_{\chi,\eta}(h_1,h_2) := \chi\left(\frac{\det h_1}{\det h_2}\right)\eta\left(\frac{1}{\det h_2}\right).$$

Then

$$\begin{aligned} &\exists\varphi,\chi,\eta \text{ such that } P_H(\varphi,\psi_{\chi,\eta}) \neq 0 \\ &\qquad \iff \pi \text{ is the functorial transfer of some } \Pi \text{ on } \mathrm{GSpin}_{2n+1}(\mathbb{A}) \qquad (1.1)\\ &\qquad \iff \pi \text{ has an } \eta\text{-Shalika model.} \qquad (1.2)\end{aligned}$$

These equivalences are a combination of results of Friedberg–Jacquet [FJ93] and Asgari–Shahidi [AS06]. I have suppressed the additive character, denoted ψ in these works.

Remark 1.1: All of these examples are *spherical pairs*. Period integrals for spherical pairs are staples of the relative Langlands program of Sakellaridis–Venkatesh [SV17].

Definition 1.2. We say (G,H) is *spherical* if H has an open orbit on $B\backslash G$, for $B \subset G$ the upper-triangular Borel subgroup. Equivalently, this is if and only if there exists $u \in G$ such that BuH is open in G.

The slogan here is that *"(G,H) is spherical if H is big in G"*.

Example 1.3: For the pair $(\mathrm{GL}_4, \mathrm{GL}_2 \times \mathrm{GL}_2)$, one may take u to be the element

$$u = \begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

This element will later be crucially important for p-adic variation.

2. Friedberg–Jacquet integrals

We now specialise to the particular case of interest. Whilst [BGW25] treats arbitrary GL_{2n}, for simplicity here I specialise further to GL_4, where the main ideas are already present.

Let, then, $G = \mathrm{GL}_4$ and $H = \mathrm{GL}_2 \times \mathrm{GL}_2$. Let π be a unitary[2] cuspidal automorphic representation of $G(\mathbb{A})$. We assume there exist φ, χ and η such that

$$P_H(\varphi, \psi_{\chi,\eta}) \neq 0,$$

whence by (1.1) we know π is the transfer of some cuspidal (generic) automorphic representation Π of $\mathrm{GSpin}_5(\mathbb{A})$. In this case, we have the exceptional isomorphism $\mathrm{GSpin}_5 \cong \mathrm{GSp}_4$, and from now on we consider Π as a representation of $\mathrm{GSp}_4(\mathbb{A})$.

Essential to our whole study is the following decomposition of P_H into a product of local zeta integrals from [FJ93].

Proposition 2.1 (Friedberg–Jacquet). *For $\varphi = \otimes_v \varphi_v$ a pure tensor, the period integral*

$$P_H(\varphi, \psi_{\chi,\eta}) = \prod_v \zeta_v(\varphi_v, \chi_v)$$

is a product of local zeta integrals.

Somehow the existence of this decomposition is more important to this note than the actual definition of ζ_v. Indeed, in the talk, I described the precise definition as 'noise' that distracts from the narrative. However, for completion let me at least sketch the definition.

[2] This is not necessary, but ensures $s = 1/2$ is the central value of $L(\pi, s)$, simplifying exposition here.

We have an intertwining map

$$\pi \longrightarrow \mathcal{S}^{\eta}(\pi) = \otimes_v \mathcal{S}^{\eta_v}(\pi_v)$$
$$\varphi \longrightarrow W_\varphi = \otimes W_v$$

into the η-Shalika model of π, which exists by (1.2). Here each W_v is a function on $\mathrm{GL}_2(\mathbb{Q}_v)$ transforming via η (and a suppressed additive character) under the Shalika subgroup. Then we define

$$\zeta_v(\varphi_v, \chi_v) := \int_{\mathrm{GL}_n(\mathbb{Q}_v)} W_v \begin{pmatrix} g & \\ & 1 \end{pmatrix} \chi_v(\det g) dg,$$

the *local Friedberg–Jacquet integral.*[3]

This proposition attains much greater importance through the following theorem.

THEOREM 2.2. (1) *(Friedberg–Jacquet [FJ93]). If $v \nmid \infty$, there exists $\varphi_v \in \pi_v$ such that*

$$\zeta_v(\varphi, \chi_v) = L(\pi_v \otimes \chi_v, \tfrac{1}{2}).$$

(2) *(Sun [Sun19]). At $v = \infty$, the non-vanishing hypothesis holds.*

COROLLARY 2.3. *There exists $\varphi \in \pi$ such that*

$$P_H(\varphi, \psi_{\chi,\eta}) = (*) \cdot L(\pi \otimes \chi, \tfrac{1}{2}),$$

where $(*)$ is a non-zero scalar.

Note $s = 1/2$ is the central value of this L-function. A more precise version of this corollary shows that $(*)$ can take two possible values $C^{\pm} \in \mathbb{C}^\times$, with the sign determined by $\chi(-1) = \pm 1$.

3. p-adic interpolation

Given a collection of numbers, it is natural to discuss its interpolation by an analytic object. In classical real analysis, for example, this is Lagrangian interpolation. In number theory, it has proved fruitful to consider p-adic analogues, for example through p-adic L-functions and Iwasawa theory, or Hida/Coleman families and eigenvarieties.

3.1. An abstract interpolation problem

Let us set up a very general interpolation problem, in the language of p-adic measures/distributions. Let

$$\Sigma_p := \Big\{\chi \text{ Dirichlet characters of conductor } p^\beta,\ \beta \geq 0\Big\}.$$

Note that any $\chi \in \Sigma_p$ is naturally a character on $\mathbb{Z}_p^\times$, lifting under the natural projection $\mathbb{Z}_p^\times \to (\mathbb{Z}/p^\beta\mathbb{Z})^\times$.

[3] In practice the integral is attached to W_v, not φ_v; whilst one may write down a global intertwining map explicitly, the local Shalika models are then characterised only up to non-zero scalar. However, we are primarily interested in vanishing versus non-vanishing of these integrals, which is not affected by the choice of local intertwinings, so this imprecision is harmless in this note.

Let

$$\{A_\chi \in \overline{\mathbb{Q}}_p : \chi \in \Sigma_p\}$$

be a collection of algebraic p-adic numbers.

Definition 3.1. We say $\{A_\chi\}$ is *p-adically interpolable* if there exists a locally analytic distribution μ (of restricted growth) such that

$$\int_{\mathbb{Z}_p^\times} \chi \cdot \mu = A_\chi.$$

By a *locally analytic distribution*, we mean an element of the continuous linear dual of locally analytic functions on $\mathbb{Z}_p^\times$. Each $\chi \in \Sigma_p$ is a locally constant, hence locally analytic, function on $\mathbb{Z}_p^\times$ by the above.

Remarks:

(i) Via p-adic Fourier theory (see e.g. [RJW25, §3.7] for a summary) one can view μ as a p-adic rigid analytic function on the open unit ball in $\mathbb{C}_p$, and thus this really obtains the flavour of a p-adic Lagrangian interpolation; but we continue to use the measure-theoretic language.

(ii) This definition is vacuous without the notion of 'sufficiently small' growth, as developed by Vishik and Amice-Velu (see e.g. [AV75]). Indeed, without it, the set Σ_p becomes discrete in the parameter space.

As an important special case, if μ has growth 0 (i.e. it is bounded), then it is a p-adic measure on $\mathbb{Z}_p^\times$. This corresponds to bounded rigid analytic functions under the identification of (i).

The definition of growth and the theory of p-adic analysis is peripheral to the present notes, so I will not discuss them further.

The following example is a (very) important example of such interpolation.

Example 3.2: Let $E/\mathbb{Q}$ be an elliptic curve with good ordinary reduction at p, and let

$$A_\chi = e_p(E,\chi)\frac{L^{(p)}(E,\chi,1)}{\Omega_E^{\pm}\cdot 2\pi i},$$

where $e_p(E,\chi)$ is a (completely explicit) modified Euler factor at p, and $L^{(p)}$ is the L-function with the factor at p removed. In [MSD74], Mazur–Swinnerton-Dyer used modular symbols to prove that $\{A_\chi\}$ is interpolable by a measure $\mu = L_p(E)$, the p-adic L-function of E.

Remarks:

- The exact factors appearing here are predicted by a (later!) general conjecture of Coates–Perrin-Riou [Coa89], which hypothesised the existence of p-adic L-functions for quite general motives. In the case of elliptic curves, the factor $e_p(E,\chi)$ here is exactly their factor; and the $2\pi i$ appears through the modified Euler factor at ∞.
- This measure has been used by Kato, Skinner-Urban et. al to control the ranks of Selmer groups and deduce special cases of the Birch and Swinnerton-Dyer conjecture. The methods go through proving Iwasawa Main Conjectures. For a detailed summary and references, see [RJW25, Appendix B].

- There has been significant work in generalising this story beyond the case of good ordinary reduction, and to modular forms of higher weights.

3.2. p-adic interpolation for GL_4

We return to our main setting. Hopefully the reader will now find the following a natural question.

QUESTION 3.3. *For π a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A})$ as above, and $\varphi \in \pi$, is the set*

$$\{P_H(\varphi, \psi_{\chi,\eta}) : \chi \in \Sigma_p\}$$

p-adically interpolable?

Henceforth we will focus on *cohomological* π (cohomology classes attached to cusp forms are natural generalisations of the modular symbols used by Mazur–Swinnerton-Dyer). In this setting, the question of p-adic interpolation was first considered by Ash–Ginzburg [AG94], and later taken up again by Dimitrov–Januszewski–Raghuram [DJR20], Gehrmann [Geh18], and Barrera–Dimitrov–Graham–Jorza–W. [BDW, BDG$^+$]. All of these works proved interpolation theorems answering the question above (and generalisations thereof) under various hypotheses, and Corollary 3.7 – the most general iteration to date, which is proved in [BDW, BDG$^+$] – builds on all of them.

First, we note a somewhat vacuous result. It is not too hard to show that:

LEMMA 3.4. *Fix $\beta \geq 1$. There exists a distribution μ_β on $\mathbb{Z}_p^\times$ such that*

$$\int_{\mathbb{Z}_p^\times} \chi \cdot \mu_\beta = (*) \cdot P_H(\varphi, \psi_{\chi,\eta})$$

for all Dirichlet characters χ of conductor exactly *p^β.*

Here $(*)$ is the same non-zero factor as we saw in Corollary 2.3.

The reader will notice, however, that this gives interpolation only at a finite number of values. In particular, by itself it's a useless result (reflected in the relative ease of the proof). Moreover, in this form it cannot be improved: the distributions μ_β *do not* fit together as we vary β.

The problem here is at p. There are already hints of this in Mazur–Swinnerton-Dyer's theorem, where one had to delete the L-factor at p in $L(E, \chi, 1)$ and instead introduce the 'modified' factor $e_p(E, \chi)$ at p. Here, too, we must make modifications at p, which will come via modifications to the local zeta integral ζ_p.

NOTATION 3.5: Let

$$t_p = \begin{pmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in G(\mathbb{Q}_p),$$

and let

$$u = \begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in G(\mathbb{Q}_p)$$

as in Example 1.3.

Let

$$Q = \left(\begin{array}{c|c} * & * \\ \hline & * \end{array}\right) \subset G$$

be the maximal standard parabolic with Levi subgroup H, let

$$J = \Big\{ g \in G(\mathbb{Z}_p) : g \pmod p \in Q(\mathbb{F}_p) \Big\} \subset G(\mathbb{Z}_p)$$

be the Q-parahoric subgroup, and let

$$U_p = [J t_p J]$$

be the Hecke operator attached to J and t_p.

THEOREM 3.6. *Let $\varphi = \otimes_v \varphi_v$ such that:*

- *$\varphi_p \in \pi_p^J$ (so π is "parahoric-spherical"),*
- *φ_p is "non-critical".*

Then there exist distributions $\mu_\beta(\varphi)$ on $\mathbb{Z}_p^\times$ such that:

(1) *we have the interpolation*

$$\int_{\mathbb{Z}_p^\times} \chi \cdot \mu_\beta(\varphi) = (*) \cdot P_H(u t_p^\beta \cdot \varphi, \psi_{\chi,\eta})$$

for all χ of conductor exactly p^β, and

(2) *We have $\mu_{\beta+1}(\varphi) = \mu_\beta(U_p\varphi)$.*

COROLLARY 3.7. *If $U_p\varphi_p = \alpha_p\varphi_p$, then $\mu := \alpha_P^{-\beta}\mu_\beta(\varphi)$ is independent of β, and we have*

$$\begin{aligned}\int_{\mathbb{Z}_p^\times} \chi \cdot \mu &= (*) \cdot \alpha_p^{-\beta} \cdot P_H(u t_p^\beta \cdot \varphi, \psi_{\chi,\eta}) \\ &= (*) \cdot \alpha_p^{-\beta} \cdot L^{(p)}\Big(\pi \otimes \chi, \tfrac{1}{2}\Big) \cdot \zeta_p\Big(u t_p^\beta \cdot \varphi_p, \chi_p\Big)\end{aligned}$$

for all[4] $\chi \in \Sigma_p$, where $p^\beta = \mathrm{Cond}(\chi)$.

So, an updated question is:

QUESTION 3.8. *What is $\zeta_p(u t_p^\beta \cdot \varphi_p, \chi_p)$?*

...in particular, for which φ_p is it non-zero?

4. Conjecture on p-refined Friedberg–Jacquet integrals

In [BGW25, Part III], we make a conjecture that would answer the non-vanishing part of Question 3.8. It is a local analogue of the global equivalence (1.1) above. Recall that (1.1)

[4]Strictly the theorem implies the result for ramified χ; but a slight modification also handles unramified χ, taking $\beta = 1$.

said that non-vanishing of the global period integral holds if and only if π is a functorial transfer of Π on $\mathrm{GSp}_4(\mathbb{A})$.

4.1. Statement of the conjecture

We have analogues of Notation 3.5 for GSp_4.

NOTATION 4.1: Let

$$Q' = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \subset \mathrm{GSp}_4$$

be the Klingen parabolic, with parahoric subgroup

$$J' = \Big\{ g \in \mathrm{GSp}_4(\mathbb{Z}_p) : g \pmod p \in Q(\mathbb{F}_p) \Big\} \subset G(\mathbb{Z}_p).$$

Let

$$t'_p = \begin{pmatrix} p^2 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{Q}_p),$$

and let

$$U'_p = [J' t'_p J']$$

be the Klingen U_p-operator.

We state our conjecture in a purely local setting. As such, let π_p be a smooth admissible irreducible representation of $\mathrm{GL}_4(\mathbb{Q}_p)$. We assume π_p is the functorial transfer of Π_p on $\mathrm{GSp}_4(\mathbb{Q}_p)$, and that π_p is Q-parahoric-spherical. If either fails, then our conjecture is empty.

DEFINITION 4.2. Let $\varphi_p \in \pi_p^{J_p}$ be a U_p-eigenvector with $U_p\varphi_p = \alpha_p\varphi_p$. We say φ_p is *transfer from* GSp_4 if α_p is a U'_p-eigenvalue on $\Pi_p^{J'_p}$.

We then conjectured the following local form of Friedberg–Jacquet.

CONJECTURE 4.3 (Barrera-Graham-W.). *Let φ_p be as in Definition 4.2, and let $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be finite order and ramified. Then*

$$\zeta_p(ut_p^\beta \cdot \varphi_p, \chi_p) \neq 0 \iff \varphi_p \text{ is transfer from } \mathrm{GSp}_4.$$

EXAMPLE 4.4: Suppose π_p is spherical and regular at p; in particular

$$\pi_p = \mathrm{Ind}_B^G\, \theta$$

is unramified principal series. Then $\dim \pi_p^J = 6$, and the regularity condition ensures

$$\pi_p^J = \bigoplus_{i=1}^{6} \mathbb{C}\cdot\varphi_i, \qquad U_p\varphi_i = \alpha_i\varphi_i$$

with the α_i pairwise distinct. In this case, precisely 4 of these 6 eigenvectors are transfer from GSp_4:

$$\begin{aligned} \varphi_1' &\longmapsto \varphi_1 \\ \varphi_2' &\longmapsto \varphi_2 \\ \varphi_3' &\longmapsto \varphi_3 \\ \varphi_4' &\longmapsto \varphi_4 \\ &\qquad \varphi_5 \\ &\qquad \varphi_6. \end{aligned}$$

Note that $\dim_{\mathbb{C}} \Pi_p^{J'} = 4$, and indeed that

$$\Pi_p^{J'} = \bigoplus_{i=1}^{4} \mathbb{C} \cdot \varphi_i'.$$

The conjecture then says that under the map

$$\begin{aligned} \pi_p &\longrightarrow \mathbb{C}, \\ \varphi &\longmapsto \zeta_p(ut_p^\beta \cdot \varphi, \chi_p), \end{aligned}$$

the eigenvectors $\varphi_1, \varphi_2, \varphi_3$, and φ_4 are mapped to non-zero numbers, whilst φ_5 and φ_6 are sent to 0.

4.2. Evidence

The first evidence we provide is again purely local.

Proposition 4.5 (Dimitrov–Januszewski–Raghuram). *If π_p is spherical and regular, then $\Leftarrow$ holds in the conjecture.*

This is [DJR20, Prop. 3.4, Lem. 3.6]. The proof is by direct computation of the zeta integral.

Our other evidence, for $\Rightarrow$, is a partial result that has a more interesting proof. It is global in nature, stemming from p-adic families through π and the geometry of the GL_{2n}-eigenvariety at classical (symplectic) points. In particular, we have the following pair of theorems.

Suppose now π is a cohomological cuspidal automorphic representation, and that π_p is parahoric-spherical. Again for simplicity let us specialise to the case GL_4 and $P = Q$ (though again we prove more general analogues).

Theorem 4.6 (Barrera–Dimitrov–W.). *Suppose that φ_p is non-critical, that π has regular weight, and that there exists a Dirichlet character χ such that*

$$\zeta_p(ut_p^\beta \cdot \varphi_p, \chi_p) \neq 0.$$

Then (π, φ_p) varies in a 1-dimensional Q-parabolic family with a Zariski-dense set of Shalika classical points.

This is proved in [BDW, Thm. B]. We call such a family a *Shalika family.*

THEOREM 4.7 (Barrera–Graham–W.). *If φ_p is not transfer from* GSp_4*, then (π, φ_p) cannot vary in any positive-dimensional Q-parabolic Shalika family.*

In particular, combining the theorems we find $\Rightarrow$ in Conjecture 4.3 holds whenever π_p is the p-component of a global π satisfying the assumptions of 4.6.

4.3. More general analogues

For simplicity, we have restricted here to the case $G = \mathrm{GL}_4$ and parabolic Q. We actually give a version for general $G = \mathrm{GL}_{2n}$, and general 'spin' standard parabolics $P \subset G$ (those symmetric under reflection in the anti-diagonal, i.e. those corresponding to parabolics $P' \subset \mathrm{GSpin}_{2n+1}$). Let us sketch the more general conjecture, and our evidence towards it.

Attached to the parabolic P is a natural diagonal matrix $t_P \in \mathrm{GL}_{2n}(\mathbb{Q}_p)$, whose entries are non-negative decreasing powers of p (with jumps corresponding to the blocks in the Levi of P). We also have a corresponding parahoric J_P, a commutative level P-Hecke algebra, and analogues t_p', J_P' etc. for GSpin_{2n+1}. We can use this to formulate a general version of an eigenvector being 'transfer from GSpin_{2n+1}'.

CONJECTURE 4.8 (B.–G.–W.). *Let $\varphi_p \in \pi_p^{J_P}$ be a Hecke eigenvector, and let $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be finite order ramified. Then*

$$\zeta_p(ut_P^\beta \cdot \varphi_p, \chi_p) \neq 0$$
$$\iff \varphi_p \text{ is transfer from } \mathrm{GSpin}_{2n+1}, \quad \text{and} \quad P \subset Q.$$

For a precise statement, see [BGW25, Conj. 8.4].

REMARK 4.9: This highlights the importance of the parabolic Q in studying the twisted integrals. This is not a surprise: the necessity that $P \subset Q$ is related to the Panchishkin condition for existence of p-adic L-functions, as explained in [BDW, Intro.].

We have the following analogue of Proposition 4.5:

PROPOSITION 4.10 (Barrera–Graham–W.). *Suppose π_p is spherical and regular. Then in Conjecture 4.8:*

- $\Leftarrow$ *holds for any eigenvector $\varphi_p \in \pi_p^{J_P}$;*
- $\big[\zeta_p(ut_P^\beta \cdot \varphi, \chi_p) \neq 0\big] \quad \Rightarrow \quad \big[P \subset Q\big]$.

Again, the proofs of these results are by (long and technical) direct computation of the zeta integral, carried out in [BGW25, §9] using crucially the unramified structure.

To finish off the conjecture in the spherical regular case, it only remains to prove that $\zeta_p \neq 0$ implies φ_p is transfer from GSpin_{2n+1}. The main results of [BGW25] are Theorems 4.10 and 4.11 *op. cit.* They are used to deduce a general form of Theorem 4.7 above, which again leads to a global proof of special cases of $\Rightarrow$ in Conjecture 4.8. For a precise statement of this application, see [BGW25, Thm. 8.9].

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